L.A. CHERNOV

WAVE PROPAGATION IN A RANDOM MEDIUM

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TRANSLATOR'S PREFACE

This book is a translation of L.A. Chernov's "Распространение Волн В Среде Со Случайными Неоднородностями", literally "Propagation of Waves in a Medium with Random Inhomogeneities". It is hoped that Chernov's book (together with the translation of V.I. Tatarski's book on wave propagation in a turbulent medium, also in this series) will furnish a comprehensive and authoritative survey of the present state of research in the field of wave propagation in random media, with special emphasis on important Russian contributions.

The translation departs from the original in one small particular: the Russian words for amplitude and for level (amplitude measured in logarithmic units) have both been rendered as amplitude, since in every case the context precludes confusion.

There is no ideal system for transliterating the Cyrillic alphabet into the Roman alphabet. Of the various systems which exist, I prefer and have used that of Prof. E.J. Simmons of Columbia University.

AUTHOR'S PREFACE

This monograph contains a systematic treatment of the theory of wave propagation in a medium with random inhomogeneities. Practical problems of acoustics, optics and radiophysics have stimulated and made urgent the need for such a treatment.

In Part I we study the problem of wave propagation using the ray approximation. In Part II we deal with the diffraction theory of wave propagation. In Part III we examine the question of how fluctuations in the incident wave affect the diffraction image formed by a focusing system; this question is of considerable interest in hydroacoustics and astronomical optics. Some theoretical deductions are compared with experimental data. (The most involved calculations are relegated to appendices.)

The author takes this opportunity to express his appreciation to L. M. Brekhovskikh, Corresponding Member of the Academy of Sciences of the U.S.S.R., for his constant interest in this book and for the valuable advice he gave while it was being written. The author recalls with gratitude the late Professor G. S. Gorelik; the latter undertook to familiarize himself with the manuscript and made numerous helpful critical remarks which led to its improvement.

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Part III

THE INFLUENCE OF FLUCTUATIONS ON THE DIFFRACTION IMAGE OF A FOCUSING SYSTEM

INTRODUCTION

Inhomogeneity is a characteristic property of every real medium. Two types of inhomogeneities are observed, regular and random. Regular inhomogeneities are produced by spatial variation of the mean characteristics of a medium, while random inhomogeneities are produced by deviations from the mean values.

Both regular and random inhomogeneities can influence (although in different ways) wave propagation in a medium. For example, such phenomena as refraction and duct propagation are caused by the influence of regular inhomogeneities, while the scattering of waves is caused by random inhomogeneities. The scattered waves are superimposed on the incident wave and lead to amplitude and phase fluctuations of the resultant field. This explains the loudness fluctuations of sound propagated in the atmosphere and the sea, the twinkling of stars, and (in part) the fading of radio signals. Regular changes in a real medium occur in some preferred direction (depth in the sea, height in the atmosphere). Therefore, as a first approximation, a real medium can be regarded as having a layered structure. The theory of wave propagation in layered media has been developed in the last ten to fifteen years in the extensive investigations of Brekhovskikh [1,2,3].

In spite of the fact that the twinkling of stars is a problem several thousand years old (it was of interest to Aristotle) the development of the theory of wave propagation in a medium with random inhomogeneities has likewise occurred mainly in the last decade and is due to a large extent to the practical requirements of hydroacoustics and radiophysics. In the early theoretical work of Krasilnikov [40,54] and Bergmann [14] the problem of wave propagation in a medium with random inhomogeneities was studied in the approximation of geometrical optics. Applying the method of small perturbations to the eikonal equation, these authors obtained formulas for the mean square fluctuations of amplitude and phase as functions of distance. An unimportant difference in the formulas for the amplitude fluctuations results from the fact that Bergmann considered a spherical wave (point source), while Krasilnikov carried out the

calculation for a plane wave.

The author [46] suggested another method of solving the problem of ray propagation in a medium with random inhomogeneities, a method based on application of the Einstein-Fokker-Kolmogorov equation. The propagation of a ray is regarded as a random process without after-effect (continuous Markov process), in which the role of time is played by the path length traversed by the ray. The angular distribution function of the rays obeys the Einstein-Fokker-Kolmogorov equation. This method permits one to determine not only the angular but also the linear displacement of the ray from its initial direction. Moreover, this can be done without requiring the displacement to be small, which is an advantage of this method over the method of small perturbations. Kharanen [5] has also used the Einstein-Fokker-Kolmogorov method in treating the problem of ray propagation in a medium with random inhomogeneities. However, his results are valid only for small angular and linear deflections of the ray, and in this limiting case can be obtained from the more general formulas of our work.

The problem of the intensity of radiation scattered by random inhomogeneities was examined in early papers using the wave approach. Pekeris [35] derived a general formula for wave scattering, assuming that as a result of fluctuations the refractive index deviates only slightly from some mean value. In acoustics it is usually necessary to take into account density fluctuations as well as refractive index fluctuations. In this book, we derive a more general formula which takes this factor into account. We also clarify the conditions under which, in the acoustic scattering problem, density fluctuations can be neglected in comparison with refractive index fluctuations, i.e., we show the range of applicability of Pekeris' formula in acoustics.

In recent years, the problem of amplitude and phase fluctuations when waves are propagated in a medium with random inhomogeneities has been studied by many investigators [12, 18, 20, 21, 22, 23, 32] from a diffraction theory point of view. In particular, it has been possible to ascertain the limits of applicability of ray theory for a medium with random inhomogeneities [12, 31, 33]. All investigators have applied the method of small perturbations to the wave equation; only Obukhov [12] has made use of Rytov's method [11], which was used by Rytov in studying the diffraction of light by an ultrasonic grating. The fluctuations calcu-

lated by this method are not restricted by the condition that they be small; this is the advantage of Rytov's method over the method of small perturbations. In the present book, the author also uses Rytov's method and, in doing so, generalizes the formulas for mean square amplitude and phase fluctuations obtained by Obukhov. Unlike Obukhov, who started with a Gaussian correlation coefficient for the refractive index, the author has succeeded in deriving general formulas which are not a consequence of any special form of the correlation coefficient.

Mean square amplitude and phase fluctuations are not enough to completely characterize the statistical properties of a wave field. The statistical properties of the fluctuations of a wave field can be more completely characterized by using correlation functions. However, the problem of correlation functions for fluctuations of the values of the basic characteristics of a wave field has not been investigated at all from a wave point of view. In this book, the author tries to fill in this important gap in the theory of wave propagation in a medium with random inhomogeneities. With this aim, we calculate: (1) the cross correlation coefficient of amplitude and phase at one receiving point, (2) the autocorrelation coefficient for amplitude (or phase) at different receiving points and (3) the time autocorrelation coefficient for amplitude (or phase). Nobody has heretofore inquired as to the nature of the cross correlation of fluctuations of amplitude and phase at a receiving point. In this book, we answer this question and show that the correlation between amplitude and phase fluctuations, which exists at small distances, vanishes at large distances.

The question of the autocorrelation of amplitude (or phase) fluctuations at different receiving points has been studied by many authors [14,6] but only in the ray approximation and only for the case where both receivers lie in a plane perpendicular to the direction of propagation of the wave (transverse autocorrelation). In this book, the problem of longitudinal and transverse autocorrelation is studied from a wave point of view. It is shown that at all distances the transverse autocorrelation between amplitude (or phase) fluctuations extends to approximately the same separation as the correlation between the random inhomogeneities of the medium itself. Moreover, it is shown that the longitudinal correlation extends over much greater separations than the transverse correlation. If the separation between receivers does not exceed the distance within which a ray treatment is suitable, the

fluctuations of amplitude (or phase) are practically completely correlated.

Since evidently the time variation of the inhomogeneities is due mostly to their motion, in this book we also examine the question of how the motion of the medium as a whole affects the time correlation properties of the wave field at a receiving point. The correlation theory developed in this book leads to the conclusion that the time autocorrelation function is practically independent of distance and depends mainly on the speed with which the observer moves. Experiments performed on ships moving in the ocean confirm this deduction [23]. The theoretically calculated mean autocorrelation curve is in good quantitative agreement with the experimental data.

Fluctuations in a wave incident on a focusing system are accompanied by fluctuations of the diffraction image. In the diffraction image we observe not only deviations of the intensity from the mean distribution, but also the mean distribution itself depends in an essential way on fluctuations in the incident wave. In this regard two problems arise in the theory of focusing systems: (1) investigation of the dependence of the mean distribution in the diffraction image on fluctuations in the incident wave and (2) investigation of the distribution of fluctuations in the diffraction image. As far as we know, neither of these questions has been studied previously. Both questions are studied in this book.

In the presence of strong fluctuations the mean distribution differs greatly from the distribution in the absence of fluctuations, if the dimensions of the diaphragm are larger than or of the same order as the correlation distance of the incident wave. The distribution curve falls off monotonically as we go away from the focus and, in the absence of fluctuations, forms a system of decreasing maxima. The distribution of fluctuations in the diffraction image coincides with the mean distribution in two limiting cases: (1) the fluctuations are small and the dimensions of the diaphragm are small compared to the correlation distance and (2) the fluctuations are large. If the fluctuations are small but the dimensions of the diaphragm are larger than or of the same order as the correlation distance, the distribution of the fluctuations does not coincide with the mean distribution. For the relative fluctuations at the focus the theory gives a result which coincides with that of Krasilnikov and Tatarski [10].

All the problems studied in this book are based on the use of the scalar wave equation, as in acoustics, but in passing we indicate the polarization corrections which must be introduced in going over to the case of electromagnetic waves.

Part I

RAY THEORY

Chapter I

STATISTICAL CHARACTERISTICS OF THE MEDIUM

In a medium with random inhomogeneities the index of refraction is a random function of the coordinates and the time. The changes in refractive index in the sea and in the atmosphere are usually caused by temperature fluctuations. In addition, salinity fluctuations in the sea and humidity fluctuations in the atmosphere can play a role. Apparently, these fluctuations are small in the majority of cases. Liebermann [4] has made an experimental study of temperature inhomogeneities in the ocean using a fast-acting thermometer mounted on a submarine. The curve of temperature fluctuations obtained by him is shown in Fig. 1.

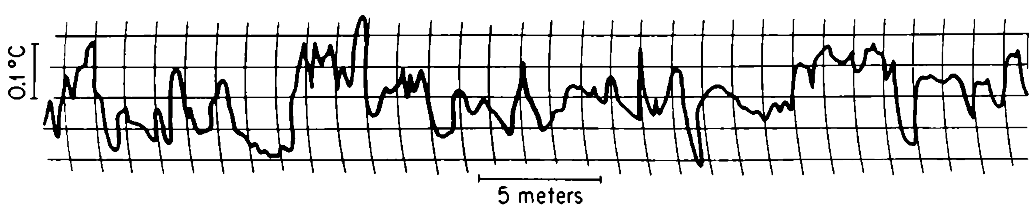

Fig. 1 Record of temperature fluctuations in the ocean. (After Liebermann)

The choice of a submarine instead of a surface vessel was dictated by the need to avoid pitching, which introduces additional errors into the measurements. The submarine moved at depths of 30 to 60 meters. At these depths the mean temperature fluctuations amounted to 0.04° C, with a ~ 60 cm as the mean size of the inhomogeneities. Such small temperature fluctuations correspond to small fractional changes in the index of refraction; the mean square fluctuation of the acoustic index of refraction is equal to $\overline{\mu^2} = 5 \times 10^{-9}$ $(\sqrt{\overline{\mu^2}} \sim 7 \times 10^{-5})$. Even if the temperature fluctuations amounted to a few degrees, the fractional changes in refractive index would still not exceed 0.01. Apparently, the influence of salinity fluctuations is even less significant.

1. The Correlation Function. We shall assume that the fluctuations in refractive index represent a random process in space and time, described by the random function of coordinates and time $\mu(x,y,z,t)$. Regarding this random process as stationary in time, we shall characterize it by the correlation function

$$N_{12} = \overline{\mu(x_1,y_1,z_1,t)\mu(x_2,y_2,z_2,t)} , \tag{1}$$

where the overbar designates averaging with respect to the time t, or, because of the ergodic hypothesis, averaging over the ensemble of realizations corresponding to the different possible states of the medium. Understanding the overbar to mean time averaging, we write

$$N_{12} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} \mu(x_1,y_1,z_1,t)\mu(x_2,y_2,z_2,t)dt . \tag{2}$$

For a spatially homogeneous process the correlation function depends only on the coordinate differences $x = x_2 - x_1$, $y = y_2 - y_1$, $z = z_2 - z_1$, i.e.

$$N_{12} = N_{12}(x,y,z) . \tag{3}$$

For $x = y = z = 0$ the function N_{12} achieves its maximum N_{11}, equal to the mean square fluctuation of refractive index $N_{11} = \overline{\mu^2}$. The correlation coefficient N is defined as the ratio of the correlation function N_{12} to the mean square fluctuation $\overline{\mu^2}$, i.e.

$$N = \frac{N_{12}}{\overline{\mu^2}} , \tag{4}$$

so that

$$N_{12} = \overline{\mu^2}\, N . \tag{5}$$

As the distance between the points is increased, the correlation coefficient decreases from its maximum value of unity and becomes small compared to unity at a distance called the correlation distance, i.e., the statistical dependence between the fluctuations disappears.

If the properties of the medium are not spatially homogeneous, then the correlation function will depend not only on the coordinate differences but also on the coordinates themselves. However, in what follows we shall consider only the statistically homogeneous case.

In addition to the three dimensional spatial correlation function (1), we introduce the four dimensional space-time correlation function

$$N_{12} = \overline{\mu(x_1,y_1,z_1,t_1)\mu(x_2,y_2,z_2,t_2)} \ . \tag{6}$$

Setting $t_2 = t_1 + t$ and taking the overbar to mean time averaging, we write

$$N_{12} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} \mu(x_1,y_1,z_1,t_1)\mu(x_2,y_2,z_2,t_1 + t)dt_1 \, . \tag{7}$$

For a stationary and spatially homogeneous random process, the four dimensional correlation function depends only on the coordinate differences and the time difference $t = t_2 - t_1$, i.e.

$$N_{12} = N_{12}(x,y,z,t) \, . \tag{8}$$

The corresponding correlation coefficient is defined by Eq. (4). As t grows the correlation coefficient decreases and becomes small compared to unity for a time $t \sim T$ called the correlation time. The three dimensional correlation function is the special case of the four dimensional one for $t_1 = t_2$.

In a brief note Fine [24] considers the problem of averaging not only in time but also in space when defining the correlation coefficient for refractive index fluctuations. However, the equivalence of these two averaging operations for a stationary and spatially homogeneous process cannot be doubted.

2. Determination of the Form of the Correlation Function. It seems that only one case can be given where the correlation function can be determined theoretically. This is the case of homogeneous isotropic turbulence. The field of temperature fluctuations caused by turbulence was investigated by Obukhov [13], who found for the mean square temperature differ-

ence at two points a law similar to the "two-thirds law". However, even if (under familiar conditions) the state of the atmosphere can be described in a satisfactory way by the theory of homogeneous isotropic turbulence, the question of whether this theory is applicable to the ocean has yet to be settled. In hydroacoustical investigations it is appropriate to start with correlation functions found empirically. Moreover, in atmospheric acoustics (as well as in optics and radiophysics) it is also appropriate to begin this way if the conditions for the applicability of the theory of homogeneous isotropic turbulence are violated for one reason or another.

The correlation coefficient for temperature fluctuations in the ocean was determined experimentally by Liebermann [4], who used a correlator which first multiplied together two temperature curves $T(x_1)$ and $T(x_1 + x)$ and then averaged them over the entire record. In Fig. 2

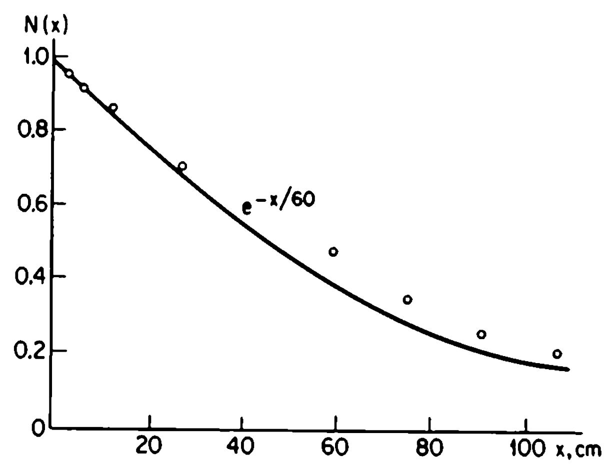

Fig. 2 Correlation coefficient of temperature fluctuations in the ocean. (After Liebermann)

the points indicate the results found by Liebermann for different distances x. The dependence of the correlation coefficient on the distance is satisfactorily approximated by the function

$$N(x) = e^{-|x|/a}, \tag{9}$$

if we set the correlation distance a = 60 cm. For a statistically isotropic medium we can write

$$N(r) = e^{-r/a} \tag{10}$$

instead of (9), where $r = \sqrt{x^2 + y^2 + z^2}$.

A peculiarity of the correlation function $e^{-|x|/a}$ should be pointed out, namely, its derivative at $x = 0$ differs from zero. This is possible only in the case where the refractive index fluctuation $\mu(x)$ is a discontinuous function. Actually, we assume the opposite, i.e., that the refractive index fluctuation $\mu(x)$ is a continuous function of x. Differentiating the product $\mu(x_1)\mu(x_1 + x)$ with respect to x, we obtain

$$\frac{d}{dx}\left[\mu(x_1)\mu(x_1 + x)\right] = \mu(x_1)\frac{d}{dx}\mu(x_1 + x) \; . \tag{11}$$

Since

$$\frac{d}{dx}\mu(x_1 + x) = \frac{d}{dx_1}\mu(x_1 + x) \; ,$$

in the right hand side of (11) we can differentiate with respect to x_1 instead of with respect to x:

$$\frac{d}{dx}\left[\mu(x_1)\mu(x_1 + x)\right] = \mu(x_1)\frac{d}{dx_1}\mu(x_1 + x) \; . \tag{12}$$

Averaging (12) with respect to the coordinate x_1 we obtain

$$\frac{dN_{12}(x)}{dx} = \overline{\mu(x_1)\frac{d}{dx_1}\mu(x_1 + x)} \; , \tag{13}$$

where

$$N_{12}(x) = \overline{\mu(x_1)\mu(x_1 + x)} = \lim_{X\to\infty}\frac{1}{2X}\int_{-X}^{+X}\mu(x_1)\mu(x_1 + x)dx_1 \; .$$

We now find the value of the derivative at x = 0:

$$\left[\frac{dN_{12}(x)}{dx}\right]_{x=0} = \overline{\mu(x_1)\,\frac{d\mu(x_1)}{dx_1}} = \overline{\frac{1}{2}\,\frac{d}{dx_1}\,\mu^2(x_1)} = \lim_{X\to\infty}\frac{1}{2X}\int_{-X}^{+X}\frac{1}{2}\,\frac{d}{dx_1}\,\mu^2(x_1)dx_1$$

$$= \lim_{X\to\infty}\frac{1}{4X}\left[\mu^2(X) - \mu^2(-X)\right] .$$

Since $\mu(x)$ is everywhere bounded, the limit of the last expression is zero, i.e.

$$\left[\frac{dN_{12}(x)}{dx}\right]_{x=0} = 0 . \tag{14}$$

The correlation coefficient $e^{-|x|/a}$ does not satisfy this condition. Consequently, the refractive index fluctuation $\mu(x)$ is a discontinuous function for this case. Actually, the temperature discontinuities are smoothed out as a result of heat conductivity, so that there is not a full correspondence between the correlation coefficient $e^{-|x|/a}$ and the actual conditions. To achieve such a correspondence the function $e^{-|x|/a}$ must be modified in such a way that the new function differs from the function $e^{-|x|/a}$ only in the immediate neighborhood of zero and has a vanishing derivative at x = 0. For example, the function

$$N(x) = \frac{a\alpha}{a\alpha - 1}\ e^{-|x|/a} - \frac{1}{a\alpha - 1}\ e^{-\alpha|x|}$$

satisfies these conditions. The smallest value of α corresponding to Liebermann's experimental data is 0.5 cm^{-1}.

The experimental data are also described in a satisfactory way by a function of the form

$$N(x) = e^{-x^2/a^2} ,$$

which is especially convenient for theoretical investigations, and has a vanishing derivative

at $x = 0$. The corresponding correlation coefficient for a statistically isotropic medium has the form

$$N(r) = e^{-r^2/a^2} \tag{15}$$

In the following we shall use the correlation coefficients (15) and (10) in studying special cases. In this regard it should not be forgotten that the correlation coefficient (10) corresponds to discontinuous changes in the refractive index fluctuations.

Chapter II

RAY STATISTICS

We turn now to a consideration of ray propagation in a medium with random inhomogeneities, assuming that a, the scale of the inhomogeneities, is large compared to the wavelength λ. In hydroacoustics this condition is often satisfied for ultrasonic waves; in the atmosphere the condition is met for light waves, since the inner dimension of the turbulent fluctuations in the atmosphere is of the order of 1 cm [49]. It should be noted that the condition that the wavelength be small compared with the scale of the inhomogeneities is only a necessary condition for the geometrical approximation to be suitable, but not a sufficient condition. If this condition is met, then the ray theory can be used in regions of linear dimension L, where L satisfies the condition $\sqrt{\lambda L} \ll a$. This condition has a simple physical meaning: the size of the first Fresnel zone for the distance in question must be small compared to the scale of the inhomogeneities. At larger distances which do not satisfy this condition the ray approximation cannot be used, and in this case diffraction theory is necessary.

In this chapter we shall restrict ourselves to the ray model and we shall assume both of the conditions $\lambda \ll a$ and $\sqrt{\lambda L} \ll a$. The necessity of the second condition will be rigorously justified later (Section 21). Moreover, we shall assume that the transit time of the ray is small compared to the characteristic scale of changes of the inhomogeneities in time.

3. <u>The Ray Equation.</u> The ray equation in the form most convenient for the considerations to follow can be obtained from Fermat's principle

$$\int_A^B \frac{d\sigma}{c} = \min . \tag{16}$$

Introducing the refractive index $n = c_0/c$, Eq. (16) can be written in the form

$$\int_A^B n(x,y,z)\,d\sigma = \min . \tag{17}$$

We shall assume that the ray trajectories belong to the family of curves expressed by the equations $x = x(u)$, $y = y(u)$, $z = z(u)$ and passing through given points A and B. The parameter u is supposed to be chosen so that it takes fixed values u_1 and u_2 at the points A and B. For example, any of the three Cartesian coordinates satisfies this condition. The arc length σ of the curve does not satisfy this condition, since it changes in going from one curve to another. Since $d\sigma = \sqrt{x'^2 + y'^2 + z'^2}\, du$ (where the prime denotes differentiation with respect to u), the Fermat principle can be written as follows:

$$\int_{u_1}^{u_2} n(x,y,z) \sqrt{x'^2 + y'^2 + z'^2}\, du = \min . \tag{18}$$

By introducing the parameter u the problem is reduced to an ordinary variational problem. Designating

$$F(x,y,z,x',y',z') \equiv n(x,y,z) \sqrt{x'^2 + y'^2 + z'^2} , \tag{19}$$

we write the Euler equations of the variational problem as

$$\frac{d}{du}\left(\frac{\partial F}{\partial x'}\right) - \frac{\partial F}{\partial x} = 0 ,$$

$$\frac{d}{du}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0 ,$$

$$\frac{d}{du}\left(\frac{\partial F}{\partial z'}\right) - \frac{\partial F}{\partial z} = 0 .$$

Because of (19) they take the form

$$\frac{d}{du}\left(\frac{nx'}{\sqrt{x'^2+y'^2+z'^2}}\right) - \sqrt{x'^2+y'^2+z'^2}\;\frac{\partial n}{\partial x} = 0 ,$$

$$\frac{d}{du}\left(\frac{ny'}{\sqrt{x'^2+y'^2+z'^2}}\right) - \sqrt{x'^2+y'^2+z'^2}\;\frac{\partial n}{\partial y} = 0 ,$$

$$\frac{d}{du}\left(\frac{nz'}{\sqrt{x'^2+y'^2+z'^2}}\right) - \sqrt{x'^2+y'^2+z'^2}\;\frac{\partial n}{\partial z} = 0 .$$

Introducing the unit vector $\vec{s}$ tangent to the ray, with components

$$s_x = \frac{dx}{d\sigma} = \frac{dx/du}{d\sigma/du} = \frac{x'}{\sqrt{x'^2+y'^2+z'^2}} ,$$

$$s_y = \frac{y'}{\sqrt{x'^2+y'^2+z'^2}} , \qquad s_z = \frac{z'}{\sqrt{x'^2+y'^2+z'^2}} ,$$

and returning to the variable σ, we rewrite Eq. (20) as

$$\frac{d(ns_x)}{d\sigma} - \frac{\partial n}{\partial x} = 0 ,$$

$$\frac{d(ns_y)}{d\sigma} - \frac{\partial n}{\partial y} = 0 ,$$

$$\frac{d(ns_z)}{d\sigma} - \frac{\partial n}{\partial z} = 0 .$$

Instead of the three ray equations (21) we can write one vector equation

$$\frac{d(n\vec{s})}{d\sigma} - \nabla n = 0 .$$

If the refractive index is given as a function of the coordinates, then Eq. (22), together with the equation $\vec{s} = d\vec{r}/d\sigma$, allows us to find the equation of the trajectories $\vec{r} = \vec{r}(\sigma)$ with given initial conditions.

4. The Ray Diffusion Coefficient. We shall assume that the index of refraction deviates only slightly from a mean value equal to unity, i.e.

$$n(x,y,z) = 1 + \mu(x,y,z) \ , \ |\mu| \ll 1 \ . \tag{23}$$

Then Eq. (22) can be rewritten as

$$\frac{d(n\vec{s})}{d\sigma} - \nabla\mu = 0 \ . \tag{24}$$

We now calculate the mean square deviation $\overline{\epsilon^2}$ of the ray from its initial direction after going a distance $\Delta\sigma$. We choose the path $\Delta\sigma$ so that it is large compared to the correlation distance of the refractive index, but so that the deviation of the ray along the path is still small. Integrating (24) along the path $\Delta\sigma$, we obtain

$$n'\vec{s}' - n\vec{s} = \int_0^{\Delta\sigma} \nabla\mu \, d\sigma \ . \tag{25}$$

It is physically clear that the angle of deviation of the ray from its initial direction is determined by the change of the refractive index along the whole path $\Delta\sigma$, and that the longer the path $\Delta\sigma$ compared to the correlation distance, the less the deviation depends on the values n and n' of the refractive index at the ends of the path. Therefore setting the random values of the refractive index at the ends of the path $\Delta\sigma$ equal to the mean value (unity) we obtain

$$\vec{s}' - \vec{s} = \int_0^{\Delta\sigma} \nabla\mu \, d\sigma \ . \tag{26}$$

Squaring both sides of (26) and bearing in mind that

$$(\vec{s}' - \vec{s})^2 = 2(1 - \vec{s}'\cdot\vec{s}) = 2(1 - \cos \epsilon) \sim \epsilon^2 ,$$

we find

$$\epsilon^2 = \int_0^{\Delta\sigma}\int_0^{\Delta\sigma} \nabla_1\nabla_2(\mu_1\mu_2)\, d\sigma_1 d\sigma_2 \,. \tag{27}$$

Averaging the correlation coefficient for the refractive index fluctuations over the ensemble of realizations of the medium, we obtain

$$\overline{\epsilon^2} = \overline{\mu^2} \int_0^{\Delta\sigma}\int_0^{\Delta\sigma} \nabla_1\nabla_2\, N(x_1 - x_2,\, y_1 - y_2,\, z_1 - z_2)\, d\sigma_1 d\sigma_2 \,. \tag{28}$$

Since the curvature of the ray is small, the integration along the ray can be replaced by integration along a straight line:

$$\overline{\epsilon^2} = \overline{\mu^2} \int_0^{\Delta\sigma}\int_0^{\Delta\sigma} \nabla_1\nabla_2\, N(x_1 - x_2,\, y_1 - y_2,\, z_1 - z_2)\, dr_1 dr_2 \,. \tag{29}$$

Introducing relative coordinates

$$x = x_1 - x_2,\; y = y_1 - y_2,\; z = z_1 - z_2,\; r = r_1 - r_2$$

and center-of-mass coordinates

$$x_o = \tfrac{1}{2}(x_1 + x_2),\; y_o = \tfrac{1}{2}(y_1 + y_2),\; z_o = \tfrac{1}{2}(z_1 + z_2),\; r_o = \tfrac{1}{2}(r_1 + r_2)\,,$$

we have

$$\nabla_1\nabla_2\, N(x_1 - x_2,\, y_1 - y_2,\, z_1 - z_2) = -\nabla^2 N(x,y,z)\,.$$

Since $\Delta\sigma \gg a$, we can integrate with respect to r from the limits $-\infty$ to $+\infty$. Eq. (29) takes the form

$$\overline{\epsilon^2} = -\overline{\mu^2} \int_0^{\Delta\sigma} dr_o \int_{-\infty}^{+\infty} \nabla^2 N\, dr \,. \tag{30}$$

Since the correlation coefficient N is an even function, we finally obtain

$$\overline{\epsilon^2} = -2\overline{\mu^2}\,\Delta\sigma \int_0^\infty \nabla^2 N\,dr . \tag{31}$$

If the medium is statistically isotropic and if consequently N depends only on $r = \sqrt{x^2 + y^2 + z^2}$, the Laplacian ∇^2 should be expressed in spherical coordinates. Eq. (31) then takes the form

$$\overline{\epsilon^2} = -2\overline{\mu^2}\,\Delta\sigma \int_0^\infty \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial N}{\partial r}\right) dr \tag{32}$$

or

$$\overline{\epsilon^2} = 4D\,\Delta\sigma , \tag{33}$$

where

$$D = -\frac{1}{2}\overline{\mu^2} \int_0^\infty \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial N}{\partial r}\right) dr \tag{34}$$

is a constant which characterizes the medium and is defined in terms of the correlation coefficient N. The quantity D plays the role of a ray diffusion coefficient.

Setting $N = e^{-r^2/a^2}$, we find

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial N}{\partial r}\right) = -\frac{6}{a^2}\,e^{-r^2/a^2} + \frac{4r^2}{a^4}\,e^{-r^2/a^2}$$

Carrying out the integration in (34), we obtain

$$D = \sqrt{\pi}\;\frac{\overline{\mu^2}}{a} . \tag{35}$$

Setting $\overline{\mu^2} = 5 \times 10^{-9}$, a = 60 cm, we find

$$D \sim 1.5 \times 10^{-10}\ \text{cm}^{-1} = 1.5 \times 10^{-5}\ \text{km}^{-1} .$$

If we use the other form of the correlation coefficient $N = e^{-r/a}$, then the integrand becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial N}{\partial r}\right) = \frac{1}{a^2} e^{-r/a} - \frac{2}{ar} e^{-r/a} .$$

Because of the presence of the factor $1/r$ in the second term, the integral (34) will diverge logarithmically at zero. There is nothing unexpected about this. The correlation coefficient $e^{-|r|/a}$ corresponds to discontinuous changes in the refractive index fluctuation μ, in which case the original ray differential equation (24) should not be used, since the gradient of refractive index fluctuations $\nabla\mu$ becomes infinite at the points of discontinuity.

5. The Angular Distribution of Rays. The Einstein-Fokker-Kolmogorov Equation. Since the properties of a medium with random inhomogeneities are given statistically, it is only possible to predict the probability $W(\theta,\phi,\sigma)$ that a ray which has traversed a path σ will have the direction defined by the angles θ and ϕ, if we are given the probability $W_o(\theta,\phi,0)$ for the initial point of the path (θ is the polar angle and ϕ the azimuth). In other words, the propagation of a ray in a medium with random inhomogeneities can be regarded as a stochastic process without after-effect (continuous Markov chain), in which the role of time is played by the path σ traversed by the ray. The distribution function $W(\theta,\phi,\sigma)$ satisfies the Einstein-Fokker-Kolmogorov equation [25]

$$Q\frac{\partial W}{\partial \sigma} = -\frac{\partial}{\partial\theta}(A_\theta QW) - \frac{\partial}{\partial\phi}(A_\phi QW) + \frac{1}{2}\frac{\partial^2}{\partial\theta^2}(B_{\theta\theta} QW) , \tag{36}$$

$$+\frac{1}{2}\frac{\partial^2}{\partial\phi^2}(B_{\phi\phi} QW) + \frac{\partial^2}{\partial\theta\partial\phi}(B_{\theta\phi} QW)$$

where

$$Q = \sin\theta , \quad A_\theta = \frac{\overline{\Delta\theta}}{\Delta\sigma} , \quad A_\phi = \frac{\overline{\Delta\phi}}{\Delta\sigma} , \tag{37}$$

$$B_{\theta\theta} = \frac{\overline{(\Delta\theta)^2}}{\Delta\sigma} , \quad B_{\phi\phi} = \frac{\overline{(\Delta\phi)^2}}{\Delta\sigma} , \quad B_{\theta\phi} = \frac{\overline{\Delta\theta\,\Delta\phi}}{\Delta\sigma} . \tag{38}$$

The coefficients A and B are easily expressed in terms of the diffusion coefficient D, if we use the following formulas of spherical trigonometry:

$$\cos(\theta + \Delta\theta) = \cos\theta \cos\epsilon + \sin\theta \sin\epsilon \cos\psi\,, \tag{39}$$

$$\frac{\sin\Delta\phi}{\sin\epsilon} = \frac{\sin\psi}{\sin(\theta + \Delta\theta)}\ . \tag{40}$$

In these formulas ψ is the angle of deviation of the ray from its initial direction. Since all directions of the deviations are equally probable

$$\overline{\sin\psi} = \overline{\cos\psi} = 0, \quad \overline{\sin^2\psi} = \overline{\cos^2\psi} = \frac{1}{2}\ .$$

Replacing the sines of small angles in (40) by the angles themselves, we obtain

$$\Delta\phi = \frac{\epsilon}{\sin\theta} \sin\psi\ . \tag{41}$$

Moreover we have

$$\overline{\Delta\phi} = \frac{\overline{\epsilon}}{\sin\theta}\ \overline{\sin\psi} = 0, \quad \overline{\Delta\phi^2} = \frac{\overline{\epsilon^2}}{\sin^2\theta}\ \overline{\sin^2\psi} = \frac{1}{2}\frac{\overline{\epsilon^2}}{\sin^2\theta}\ . \tag{42}$$

Expanding (39) in powers of the small angles $\Delta\theta$ and ϵ and retaining in the expansion only terms of the second order in $\Delta\theta$ and ϵ, we obtain

$$\Delta\theta^2 + 2\tan\theta\ \Delta\theta + 2\epsilon\tan\theta\cos\psi - \epsilon^2 = 0.$$

The solution of this equation has the form

$$\Delta\theta = -\tan\theta + \sqrt{\tan^2\theta - (2\epsilon\tan\theta\cos\psi - \epsilon^2)}$$

(The second solution of the quadratic equation is not suitable since it gives a value for $\Delta\theta$ which is different from zero for $\epsilon = 0$.) Expanding the square root in powers of ϵ up to the second power, we obtain

$$\Delta\theta = -\epsilon\cos\psi + \frac{1}{2}\epsilon^2 \cot\theta\,(1 - \cos^2\psi)\,. \tag{43}$$

From this we get

$$\overline{\Delta\theta} = \frac{1}{4}\,\overline{\epsilon^2}\cot\theta\ , \quad \overline{\Delta\theta^2} = \frac{1}{2}\,\overline{\epsilon^2}\ . \tag{44}$$

Moreover, it follows from (41) and (43) that

$$\overline{\Delta\phi . \Delta\theta} = 0\ . \tag{45}$$

Using Eqs. (33), (37), (38), (42), (44) and (45), we finally obtain the following formulas for the coefficients A and B:

$$\begin{aligned} &A_\theta = \frac{1}{4}\,\frac{\overline{\epsilon^2}}{\Delta\sigma}\cot\theta = D\cot\theta\ , \quad A_\phi = 0\ , \\ &B_{\theta\theta} = 2D\ , \quad B_{\phi\phi} = \frac{2D}{\sin^2\theta}\ , \quad B_{\theta\phi} = 0\ . \end{aligned} \tag{46}$$

Then Eq. (36) takes the form

$$\sin\theta\ \frac{\partial W}{\partial\sigma} = D\,\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial W}{\partial\theta}\right) + \frac{D}{\sin\theta}\ \frac{\partial^2 W}{\partial\phi^2}\ , \tag{47}$$

which coincides with the equation for the rotational motion of Brownian particles [26]. For a ray starting out in the direction of the polar axis, Eq. (47) becomes

$$\sin\theta\,\frac{\partial W}{\partial\sigma} = D\,\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial W}{\partial\theta}\right)\ . \tag{48}$$

The non-negative solution of Eq. (48), which satisfies the condition of normalization of probability and which reduces to zero everywhere for $\sigma = 0$, $\theta \neq 0$, has the form [27]

$$W(\sigma.\theta) = \frac{1}{4\pi}\sum_{n=0}^{\infty}(2n+1)\,P_n(\cos\theta)\,e^{-n(n+1)D\sigma}\ ,$$

where $P_n(\cos\theta)$ are the Legendre polynomials.

We can form a more intuitive picture of the ray propagation if we calculate, for example, the mean value of the cosine of the angle of deviation of the ray from its initial direction:

$$\nu = \overline{\cos\,\theta} = 2\pi \int_0^\pi \cos\,\theta\,\sin\,\theta\,W d\theta\ .$$

Multiplying both sides of Eq. (48) by $2\pi\cos\theta$ and integrating with respect to θ from 0 to π, we obtain $d\nu/d\sigma$ on the left hand side. The right hand side can be transformed in the following way by integrating by parts

$$2\pi D \int_0^\pi \cos\,\theta \frac{\partial}{\partial\theta}\left(\sin\,\theta \frac{\partial W}{\partial\theta}\right) d\theta = 2\pi D \Big|_0^\pi \cos\,\theta\,\sin\,\theta \frac{\partial W}{\partial\theta}$$

$$+ 2\pi D \int_0^\pi \sin^2\theta \frac{\partial W}{\partial\theta}\,d\theta = 2\pi D \Big|_0^\pi \sin^2\theta\cdot W$$

$$- 4\pi D \int_0^\pi \sin\,\theta\,\cos\,\theta\cdot W\,d\theta = -\,2D\nu\ .$$

For ν we get the equation

$$\frac{d\nu}{d\sigma} = -\,2D\nu\ ,$$

whence we find

$$\nu = e^{-2D\sigma}\ . \tag{49}$$

At the initial point $\nu = 1$. As $\sigma \to \infty$, $\nu \to 0$, which has a simple physical meaning: after the ray has traversed a sufficiently long path σ, all of its directions become equally probable. Similarly for the mean square of the cosine we obtain

$$\overline{\cos^2\theta} = \frac{1}{3}\left(1 + 2e^{-6D\sigma}\right)\ . \tag{50}$$

In the case of small deviation angles ($\sin\theta \sim \theta$) Eq. (48) takes the form

$$\frac{\partial W}{\partial \sigma} = D\,\frac{1}{\theta}\,\frac{\partial W}{\partial \theta} + D\,\frac{\partial^2 W}{\partial \theta^2} \ .$$

The non-negative solution of this equation which satisfies the normalization condition has the simple form

$$W = \frac{1}{4\pi\ D\sigma}\ e^{-\theta^2/4D\sigma} \quad ,$$

i.e., for small deviation angles the angular distribution is Gaussian. For the mean square deviation angle we obtain

$$\overline{\theta^2} = 4D\sigma \ . \tag{51}$$

Eq. (51) determines the fluctuation of the angle of arrival of the ray. Evaluating this fluctuation for a distance σ = 1 km, and using Liebermann's data ($\overline{\mu^2} = 5 \times 10^{-9}$, a = 60 cm, $D \sim 1.5 \times 10^{-5}$ km^{-1}), we obtain $\sqrt{\overline{\theta^2}} \sim 26.5'$ (minutes of arc).

Kolchinski [28] analyzed data gathered from astronomical observations of fluctuations of the angle of arrival of rays from stars as a function of zenith distance. His analysis confirmed the relation (51) derived above.

6. The Mean Ray Displacement. We now determine the mean square rectilinear distance r from the initial point of a ray to the point at which it arrives after traversing a complicated path σ in the medium. Using the formulas

$$x = \int_0^\sigma \cos\theta\, d\sigma\ ,\ y = \int_0^\sigma \sin\theta\,\cos\phi\, d\sigma\ ,\ z = \int_0^\sigma \sin\theta\,\sin\phi\, d\sigma, \tag{52}$$

we find

$$\overline{r^2} = \overline{x^2} + \overline{y^2} + \overline{z^2} \tag{53}$$

$$= \int_0^\sigma\int_0^\sigma \left\{ \overline{\cos\theta_2 \cos\theta_1 + \sin\theta_1 \sin\theta_2 \cos(\phi_2 - \phi_1)} \right\} d\sigma_1 d\sigma_2 \, .$$

The expression in curly brackets is equal to $\overline{\cos\epsilon}$, where ϵ is the angle between the two directions (θ_2,ϕ_2) and (θ_1,ϕ_1). Therefore (53) takes the form

$$\overline{r^2} = \int_0^\sigma\int_0^\sigma \overline{\cos\epsilon} \;\; d\sigma_1 d\sigma_2 \, . \tag{54}$$

The value of $\overline{\cos\epsilon}$ can be obtained from (49). If $\sigma_2 > \sigma_1$, the deviation ϵ from the initial direction (θ_1,ϕ_1) is given by the formula

$$\overline{\cos\epsilon_{21}} = e^{-2D(\sigma_2-\sigma_1)} \quad . \tag{55}$$

If $\sigma_2 < \sigma_1$, the direction (θ_2,ϕ_2) will be the initial direction, and the deviation from it is given by the formula

$$\overline{\cos\epsilon_{12}} = e^{-2D(\sigma_1-\sigma_2)} \quad . \tag{56}$$

Taking this into consideration, we transform (54) as follows:

$$\overline{r^2} = \int_0^{\sigma} d\sigma_1 \int_{\sigma_1}^{\sigma} \overline{\cos \epsilon_{21}} \; d\sigma_2 + \int_0^{\sigma} d\sigma_1 \int_0^{\sigma_1} \overline{\cos \epsilon_{12}} \; d\sigma_2$$

$$= \int_0^{\sigma} d\sigma_1 \int_{\sigma_1}^{\sigma} e^{-2D(\sigma_2 - \sigma_1)} \, d\sigma_2 + \int_0^{\sigma} d\sigma_1 \int_0^{\sigma_1} e^{-2D(\sigma_1 - \sigma_2)} \, d\sigma_2$$

$$= \int_0^{\sigma} \frac{1}{2D} (1 - e^{-2D(\sigma - \sigma_1)}) \, d\sigma_1 + \int_0^{\sigma} \frac{1}{2D} (1 - e^{-2D\sigma_1}) \, d\sigma_1 \; .$$

Carrying out the integration, we finally obtain

$$\overline{r^2} = \frac{\sigma}{D} - \frac{1}{2D^2} (1 - e^{-2D\sigma}) \; . \tag{57}$$

For the case of small $D\sigma$ ($D\sigma \ll 1$) (57) becomes

$$\overline{r^2} = \sigma^2 (1 - \frac{2}{3} D\sigma) \; , \tag{58}$$

which coincides with the formula obtained by Smoluchowski [29] for the mean square displacement of a heavy molecule which has traversed a complicated path σ in a light gas.

If the x axis is taken along the initial direction of the ray, the mean square ray displacement in the direction of this axis is given by the formula

$$\overline{x^2} = \int_0^{\sigma}\int_0^{\sigma} \overline{\cos \theta_2 \cos \theta_1} \; d\sigma_1 d\sigma_2 \; , \tag{59}$$

as follows from (52). We transform the right hand side of (59) as follows:

$$\overline{x^2} = \int_0^{\sigma} d\sigma_1 \int_{\sigma_1}^{\sigma} \overline{\cos\theta_2 \cos\theta_1}\ d\sigma_2 + \int_0^{\sigma} d\sigma_1 \int_0^{\sigma_1} \overline{\cos\theta_2 \cos\theta_1}\ d\sigma_2 . \tag{60}$$

Eliminating $\cos\theta_2$ in the first term by using the (already encountered) formula

$$\cos\theta_2 = \cos\theta_1 \cos\epsilon_{21} + \sin\theta_1 \sin\epsilon_{21} \cos\psi ,$$

eliminating $\cos\theta_1$ in the second term by using the formula

$$\cos\theta_1 = \cos\theta_2 \cos\epsilon_{12} + \sin\theta_2 \sin\epsilon_{12} \cos\psi ,$$

and bearing in mind that $\overline{\cos\psi} = 0$, we obtain

$$\overline{x^2} = \int_0^{\sigma} d\sigma_1 \int_{\sigma_1}^{\sigma} \overline{\cos^2\theta_1 \cos\epsilon_{21}}\ d\sigma_2 + \int_0^{\sigma} d\sigma_1 \int_0^{\sigma_1} \overline{\cos^2\theta_2 \cos\epsilon_{12}}\ d\sigma_2 . \tag{61}$$

Designating the first double integral by I_1, we substitute for $\cos^2\theta_1$ and $\cos\epsilon_{21}$ using (50) and (55). This gives

$$I_1 = \int_0^{\sigma} \tfrac{1}{3}(1 + 2e^{-6D\sigma_1})d\sigma_1 \int_{\sigma_1}^{\sigma} e^{-2D(\sigma_2 - \sigma_1)} d\sigma_2 \tag{62}$$

$$= \frac{1}{6D} \int_0^{\sigma} (1 + 2e^{-6D\sigma_1})(1 - e^{-2D(\sigma - \sigma_1)})d\sigma_1 = \frac{1}{6D}\left(\sigma - \frac{1}{6D} + \frac{1}{6D} e^{-6D\sigma}\right) .$$

Similarly, we evaluate I_2, the second double integral in (61):

$$I_2 = \int_0^{\sigma} d\sigma_1 \int_0^{\sigma_1} \frac{1}{3} (1 + 2e^{-6D\sigma_2})\, e^{-2D(\sigma_1 - \sigma_2)}\, d\sigma_2$$

$$= \frac{1}{6D} \int_0^{\sigma} (1 - e^{-6D\sigma_1}) d\sigma_1 = \frac{1}{6D} (\sigma - \frac{1}{6D} + \frac{1}{6D} e^{-6D\sigma}) .$$

Finally we obtain

$$\overline{x^2} = \frac{1}{3D} \left(\sigma - \frac{1}{6D} (1 - e^{-6D\sigma})\right) .$$

Eq. (64) takes the form

$$\overline{x^2} = \sigma^2 (1 - 2D\sigma)$$

in the case of small $D\sigma$. We can now evaluate the mean square displacement of the ray initial direction. Using (57) and (64) we obtain

$$\overline{\rho^2} = \overline{r^2} - \overline{x^2} = \frac{2}{3} \frac{\sigma}{D} - \frac{1}{2D^2} (1 - e^{-2D\sigma}) + \frac{1}{18D^2} (1 - e^{-6D\sigma}) .$$

In the case of small $D\sigma$

$$\sqrt{\overline{\rho^2}} = \frac{2}{\sqrt{3}} D^{1/2} \sigma^{3/2} .$$

Using the "three-halves law", we can easily calculate the mean square displacement of a ray which has traversed a path σ in the medium. Setting $D = 1.5 \times 10^{-5}\ \mathrm{km}^{-1}$, we find

$$\sqrt{\overline{\rho^2}} \sim 4.5\mathrm{m}$$

for a distance of 1 km.

Kharanen [5] also used the Einstein-Fokker-Kolmogorov equation in studying the problem of ray propagation in a medium with random inhomogeneities. He determined the ray deviation (angular and linear) from the plane containing the initial ray direction. It is easy to show that the differential equation obtained by him is valid only for small deviations and that it can be obtained from our differential equation in this limiting case.

If we orient a spherical coordinate system so that the initial ray direction lies in the equatorial plane, then the ray direction with respect to the equator at any subsequent instant of time will be defined by the latitude X, connected with the polar angle θ by the relation $X + \theta = \pi/2$. To find the distribution of the rays in latitude, we must integrate the distribution function $W(\theta,\phi)$ with respect to azimuth ϕ. Multiplying (47) by $d\phi$ and integrating, we obtain

$$\sin\theta \frac{\partial W'}{\partial \sigma} = D \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial W'}{\partial \theta}\right) + \frac{D}{\sin\theta} \Big|_0^{2\pi} \frac{\partial W}{\partial \phi} ,$$

where

$$W'(\theta) = \int_0^{2\pi} W(\theta,\phi)\, d\phi .$$

Because of the periodicity of the function W the last term vanishes, and the equation takes the form

$$\sin\theta \frac{\partial W'}{\partial \sigma} = D\cos\theta \frac{\partial W'}{\partial \theta} + D\sin\theta \frac{\partial^2 W'}{\partial \theta^2} .$$

If the deviation of the ray from the equatorial plane ($\Theta = \pi/2$) is small, then as an approximation we can set $\sin \Theta = 1$ and $\cos \Theta = 0$. Then we get the equation

$$\frac{\partial W'}{\partial \sigma} = D \frac{\partial^2 W'}{\partial \Theta^2}$$

or the equation

$$\frac{\partial W'}{\partial \sigma} = D \frac{\partial^2 W'}{\partial x^2}$$

which Kharanen used in his work.

7. Fluctuations of the Transit Time and Intensity of the Rays. As conditions in the medium change, the time taken for a ray to go a given distance changes; moreover, the ray tube is deformed, which leads to intensity fluctuations. Let us consider the problem of transit time fluctuations and intensity fluctuations, assuming that the deviation of the rays from their initial direction (taken along the x axis) is small.

According to (16) the time taken to go the distance L is given by the formula

$$t = \frac{1}{c_o} \int_0^L n(x,y,z)dx ,$$

where the values of the refractive index $n(x,y,z)$ are taken along the ray. The mean transit time is

$$\bar{t} = \frac{1}{c_o} \int_0^L \overline{n(x,y,z)}\, dx = \frac{1}{c_o} \int_0^L dx ,$$

since $\bar{n} = 1$. The deviation from the mean is

$$\Delta t = t - \bar{t} = \frac{1}{c_o} \int_0^L n(x,y,z)dx - \frac{1}{c_o} \int_0^L dx = \frac{1}{c_o} \int_0^L \mu(x,y,z)dx,$$

and finally the mean square transit time fluctuation is

$$\overline{\Delta t^2} = \frac{\overline{\mu^2}}{c_o^2} \int_o^L dx_1 \int_o^L N(x_2 - x_1, y_2 - y_1, z_2 - z_1)\, dx_2 .$$

If $L \gg a$, then introducing relative coordinates $x = x_2 - x_1$, $y = y_2 - y_1$, $z = z_2 - z_1$, we are justified in integrating with respect to x from the limits $-\infty$ to ∞. Since $N(x,y,z)$ is an even function, we obtain:

$$\overline{\Delta t^2} = \frac{2\overline{\mu^2}L}{c_o^2} \int_o^\infty N(x,y,z)\, dx .$$

The correlation coefficient $N(x,y,z)$ differs from zero in the region where x has values of order a ($x \sim a$). Since the inclination of the ray is small, we have $y \ll a$ and $z \ll a$ in the region where the values of the correlation coefficient are appreciable. Therefore, in the last formula we can simply set $y = z = 0$. Then we obtain

$$\overline{\Delta t^2} = \frac{2\overline{\mu^2}L}{c_o^2} \int_o^\infty N(x,0,0)\, dx . \tag{68}$$

From this we can immediately find the phase fluctuation $S' = \omega\Delta t$:

$$\overline{S'^2} = 2\overline{\mu^2}\, k^2 L \int_o^\infty N(x,0,0)\, dx , \tag{69}$$

where k is the wave number.

The change of intensity is determined by the change of cross section of the ray tube. The relative change of cross section dS/S of the ray tube along the path dx is given by the formula

$$dS/S = \operatorname{div} \vec{s}\, dx \, . \tag{70}$$

Since the deviation of the ray from the x axis is small, we have

$$\operatorname{div} \vec{s} = \frac{\partial s_y}{\partial y} + \frac{\partial s_z}{\partial z} \; , \tag{71}$$

and the ray equations (21) can be rewritten as follows:

$$\frac{d(ns_y)}{dx} - \frac{\partial \mu}{\partial y} = 0 \, ,$$

$$\frac{d(ns_z)}{dx} - \frac{\partial \mu}{\partial z} = 0 \, .$$

Integrating these equations, we find

$$s_y = \frac{1}{n} \int_0^x \frac{\partial \mu}{\partial y} \, dx' \, ,$$

$$s_z = \frac{1}{n} \int_0^x \frac{\partial \mu}{\partial z} \, dx' \, .$$

Neglecting terms of the second order with respect to μ, we obtain

$$s_y(x,y,z) = \int_0^x \frac{\partial\mu(x',y,z)}{\partial y}\, dx' \quad ,$$

$$s_z(x,y,z) = \int_0^x \frac{\partial\mu(x',y,z)}{\partial z}\, dx' \quad . \tag{72}$$

Using (72) to replace s_y and s_z in (70), we find

$$\frac{dS}{S} = dx \int_0^x \nabla^2\mu\, dx' \quad , \tag{73}$$

where ∇^2 designates the transverse Laplacian

$$\nabla^2 = \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \quad .$$

The relative change of intensity is connected with the relative change of cross section of the ray tube by the simple relation

$$\frac{dI}{I} = - \frac{dS}{S} \quad . \tag{74}$$

Using (73) we find that the relative change of intensity along the path dx is

$$\frac{dI}{I} = - dx \int_0^x \nabla^2\mu\, dx' \quad .$$

To determine the change of intensity along the path L, we must integrate the last equation with respect to x between the limits 0 and L. Then we obtain

$$\log \frac{I}{I_o} = -\int_o^L dx \int_o^x \nabla^2\mu \, dx' \quad .$$

Setting $u = \int_o^x \nabla^2\mu \, dx'$ and $dv = dx$, we integrate (75) by parts. This gives

$$\log \frac{I}{I_o} = -\int_o^L (L - x) \nabla^2\mu \, dx \, .$$

Replacing the intensity ratio by the ratio of squared amplitudes, we obtain

$$\log \frac{A}{A_o} = -\frac{1}{2} \int_o^L (L - x) \nabla^2\mu \, dx \, .$$

We can use the last formula to determine the mean square amplitude fluctuations. Squaring (77) and averaging, we get

$$\overline{\left(\log \frac{A}{A_o}\right)^2} = \frac{1}{4} \int_o^L \int_o^L (L - x_1)(L - x_2) \times$$

$$\times \nabla_1^2 \nabla_2^2 N_{12}(x_2 - x_1, y_2 - y_1, z_2 - z_1) \, dx_1 dx_2$$

Going over to relative coordinates $x = x_2 - x_1$, $y = y_2 - y_1$, $z = z_2 - z_1$ and center of mass coordinates $x_o = \frac{1}{2}(x_1 + x_2)$, $y_o = \frac{1}{2}(y_1 + y_2)$, $z_o = \frac{1}{2}(z_1 + z_2)$, we can write

$$\overline{\left(\log \frac{A}{A_o}\right)^2} = \frac{1}{4} \int_o^L dx_o \int_{-\infty}^{+\infty} (L^2 - 2Lx_o + x_o^2 - \frac{x^2}{4}) \times$$

$$\times \nabla^2 \nabla^2 N_{12}(x,y,z) \, dx$$

in the case $L >> a$. Integrating with respect to x_o, we obtain

$$\overline{\left(\log \frac{A}{A_o}\right)^2} = \frac{1}{4} \int_{-\infty}^{+\infty} \left(\frac{L^3}{3} - \frac{Lx^2}{4}\right) \nabla^2\nabla^2 \; N_{12}(x,y,z) \; dx \, . \tag{80}$$

In the region of appreciable values of the correlation function we have $x \sim a$, so that the second term in brackets $Lx^2/4$ is small compared with the first term $L^3/3$. Neglecting the second term, we get

$$\overline{\left(\log \frac{A}{A_o}\right)^2} = \frac{1}{6} \overline{\mu^2} \, L^3 \int_0^\infty \nabla^2\nabla^2 \, N \, dx \; .$$

Because of the smallness of the inclination of the ray the integrand in this formula can be evaluated for $y = z = 0$. Thus we finally obtain

$$\overline{\left(\log \frac{A}{A_o}\right)^2} = \frac{1}{6} \overline{\mu^2} \, L^3 \int_0^\infty \left[\nabla^2\nabla^2 \, N\right]_{y=z=0} dx \; . \tag{81}$$

Eqs. (69) and (81) for the mean square phase fluctuation and the mean square amplitude fluctuation are not restricted by the requirement that the phase and amplitude fluctuations be small. In this respect they differ from the corresponding formulas obtained by Krasilnikov [6], who applied the method of small perturbations to the eikonal equation and to the equation expressing the conservation of energy flow in a ray tube. Using the same method of small perturbations, Bergmann [14] studied the problem of phase and amplitude fluctuations in a spherical wave (point source). The formula obtained by him differs from (81) in that it has the full Laplacian under the integral instead of the transverse Laplacian, and the numerical factor is 1/15 instead of 1/6.

Setting $N = \exp\left[-(x^2 + y^2 + z^2)/a^2\right]$, we obtain

$$\left[\nabla^2\nabla^2 \, N\right]_{y=z=0} = \frac{32}{a^4} \; e^{-x^2/a^2}$$

and

$$\int_{0}^{\infty} \left[\nabla^2 \nabla^2 N\right]_{y=z=0} dx = \frac{16\sqrt{\pi}}{a^3} .$$

Eq. (81) takes the form

$$\overline{\left(\log \frac{A}{A_o}\right)^2} = \frac{8\sqrt{\pi}}{3} \overline{\mu^2} \frac{L^3}{a^3} .$$

If we set $N = e^{-r/a}$, then

$$\left[\nabla^2 \nabla^2 N\right]_{y=z=0} = 8 \left(\frac{1}{a^2 x^2} + \frac{1}{ax^3}\right) e^{-x/a} ,$$

and the integral

$$\int_{0}^{\infty} \left(\frac{1}{a^2 x^2} + \frac{1}{ax^3}\right) e^{-x/a} dx$$

diverges at the point $x = 0$. As already noted above, the correlation coefficient $e^{-r/a}$ corresponds to discontinuous changes of the refractive index, a case in which we cannot use the ray differential equation.

Part II

DIFFRACTION THEORY

Chapter III

THE WAVE EQUATION

8. Derivation of the Wave Equation for an Inhomogeneous Medium. The wave equation for an inhomogeneous medium differs from the wave equation for a homogeneous medium by the presence of an additional term [55]. Accordingly, we do not think it amiss to give the derivation of the wave equation for an inhomogeneous medium. We shall assume that the inhomogeneous liquid or gaseous medium is in a state of equilibrium. In the absence of the force of gravity (we neglect its effect) the condition for equilibrium is that the pressure be the same at all points of the medium. Thus, inhomogeneity of the medium will be caused by spatial change of the temperature and density of the medium; these changes cannot be regarded as independent, for, since the pressure is constant, a change in density is completely determined by a change in temperature. Strictly speaking, a medium which is inhomogeneous with respect to temperature cannot be in a state of equilibrium, since temperature inhomogeneities are levelled out as a result of heat conduction. However, in weakly conducting media, the levelling out proceeds slowly and heat conduction can be neglected in considering relatively rapid acoustic processes. Under actual conditions, in the ocean and atmosphere, the temperature inhomogeneities are carried along by the flow and undergo convective motion due to the action of gravity. However, in the cases which interest us, the velocities of flow and convection are slow compared to the velocity of sound, and therefore the spatial distribution of inhomogeneities can be regarded as quasi-static. The quasi-static condition will be discussed in detail in Section 26.

Let p_o denote the pressure, which is constant at all points of space, and let $\rho_o = \rho_o(x,y,z)$ denote the density, which varies in space. Under the influence of an acoustic wave, these quantities suffer changes and take the values p and ρ, respectively, so that

$$p = p_o + p_1 \; ; \; \rho = \rho_o + \rho_1 \; , \tag{1}$$

where p_1 is the acoustic pressure and ρ_1 is the density change caused by the wave. The total pressure p, the total density ρ and the velocity of the acoustic oscillations $\vec{v}$ satisfy the hydrodynamical equations

$$\rho \frac{d\vec{v}}{dt} = -\nabla p \quad , \tag{2}$$

$$\frac{\partial \rho}{\partial t} = - \operatorname{div}(\rho\vec{v}) \quad . \tag{3}$$

Substituting for p and ρ in these equations using (1), and regarding (as usual) the acoustic perturbations p_1, ρ_1 and $\vec{v}$ as being small quantities of the first order, we discard terms of the second and higher orders of smallness. Then Eqs. (2) and (3) can be rewritten as follows:

$$\rho_o \frac{\partial \vec{v}}{\partial t} = - \nabla p_1 \, , \tag{4}$$

$$\frac{\partial \rho_1}{\partial t} = -\operatorname{div}(\rho_o \vec{v}) \quad . \tag{5}$$

Differentiating (5) with respect to time, we obtain

$$\frac{\partial^2 \rho_1}{\partial t^2} = -\operatorname{div}(\rho_o \frac{\partial \vec{v}}{\partial t}) \quad ,$$

since ρ_o does not depend on time. Substituting for $\rho_o \frac{d\vec{v}}{dt}$ in the last equation using (4), we obtain

$$\frac{\partial^2 \rho_1}{\partial t^2} = \nabla^2 p_1 \quad . \tag{6}$$

To find one more relation between p_1 and ρ_1, we must make an assumption about the thermodynamic character of the acoustic process. Considering the propagation of sound to be an adiabatic process, we can write

$$\frac{dp}{dt} = c^2 \frac{d\rho}{dt} \quad , \tag{7}$$

where c is the adiabatic sound velocity at the given point of the medium. The equation contains total rather than partial time derivatives since the relation (7) must be satisfied for a given particle of the medium and not for a given point of space. Using (1) to substitute for the pressure p and the density ρ in (7), we obtain

$$\frac{1}{c^2} \frac{\partial p_1}{\partial t} = \frac{\partial \rho_1}{\partial t} + \vec{v} \cdot \nabla \rho_o \quad , \tag{8}$$

with accuracy up to small quantities of the first order. In the case of a homogeneous medium, the convective term $\vec{v} \cdot \nabla \rho_o$ in the last equation vanishes, and therefore the relation (7) can be written for p_1 and ρ_1 using either total or partial derivatives with respect to time. In an inhomogeneous medium the relation (7) can only be written using total derivatives. In going over to partial derivatives an additional convective term $\vec{v} \cdot \nabla \rho_o$ of the first order of smallness appears. The appearance of an extra term in the wave equation is the result of this convective term.

Differentiating (8) again with respect to time, we can write it in the form

$$\frac{\partial^2 \rho_1}{\partial t^2} + \frac{\partial \vec{v}}{\partial t} \cdot \nabla \rho_o = \frac{1}{c^2} \frac{\partial^2 p_1}{\partial t^2} \quad . \tag{9}$$

If we use (6) to substitute for the first term in the left hand side of (9) and (4) to substitute for $\partial \vec{v}/\partial t$, we obtain

$$\frac{1}{c^2} \frac{\partial^2 p_1}{\partial t^2} - \nabla^2 p_1 + \nabla \log \rho_o \cdot \nabla p_1 = 0 \ . \tag{10}$$

Thus, the additional term

$$\nabla \log \rho_o \cdot \nabla p_1$$

has appeared in the wave equation. Later we shall evaluate the order of magnitude of the additional term and explain the conditions under which it can be neglected.

9. <u>The Wave Equation for a Multi-Component Medium.</u> In the preceding section we assumed that the medium is homogeneous in its composition. However, in practice, we can encounter the case where the composition of the medium changes from point to point, for example, ocean water with variable salinity or air with variable water vapor concentration. In the equilibrium state, where the pressure is the same at all points, a spatial change of density is caused by a change of temperature and composition. In other words, the density is a function of the temperature and the concentrations of the separate components which make up the medium.

We restrict ourselves to the simplest case of a medium consisting of two components. The generalization to the case of a multi-component medium is straightforward. Let ρ be the density of the medium, ρ' the density of the solvent, ρ'' the density of the dissolved component, and C the concentration, i.e.

$$C = \frac{\rho''}{\rho'} , \quad \rho = \rho' + \rho'' = \rho'(1 + C) . \tag{11}$$

The density of the solvent satisfies the equation of continuity [30]

$$\frac{\partial \rho'}{\partial t} = -\mathrm{div}(\rho' \vec{v}) , \tag{12}$$

where $\vec{v}$ is the velocity of the acoustic oscillations. A change in density of the dissolved component can be caused not only by convection but also by diffusion, i.e.

$$\frac{\partial \rho''}{\partial t} = -\mathrm{div}(\rho'' \vec{v} + \vec{I}) , \tag{13}$$

where

$$\vec{I} = -\rho' D_1 \nabla C - \rho' D_2 \nabla T \ .$$

Here D_1 is the diffusion coefficient and D_2 is the coefficient of thermal diffusion. Adding (12$'$) and (13), we obtain

$$\frac{\partial \rho}{\partial t} = -\mathrm{div}(\rho \vec{v}) - \mathrm{div}\ \vec{I} \ , \tag{14}$$

for the total density ρ of the medium. In the absence of sound, this equation takes the form

$$\frac{\partial \rho_o}{\partial t} = -\mathrm{div}\ \vec{I}_o \ ,$$

where $\vec{I}_o$ is the flow caused by diffusion and thermal diffusion in the unperturbed medium. We shall neglect slow processes (and also the conduction, flow and convection of heat) and consider the distribution of inhomogeneities to be quasi-static, i.e. ρ_o to be independent of time. Setting $\rho = \rho_o(x,y,z) + \rho_1$, $\vec{I} = \vec{I}_o + \vec{I}_1$ in (14) and restricting ourselves to small quantities of the first order in ρ_1, $\vec{v}$ and $\vec{I}_1$, we obtain

$$\frac{\partial \rho_1}{\partial t} = -\mathrm{div}(\rho_o \vec{v}) - \mathrm{div}\ \vec{I}_1 \ . \tag{15}$$

Finally, neglecting irreversible acoustic processes caused by diffusion and thermal diffusion, we write Eq. (15) in the form

$$\frac{\partial \rho_1}{\partial t} = -\mathrm{div}(\rho_o \vec{v}) \ . \tag{16}$$

Moreover, neglecting viscosity and heat conductivity, respectively, we can write the two additional equations

$$\rho_o \frac{\partial \vec{v}}{\partial t} = -\nabla p_1 \ , \tag{17}$$

$$\frac{dp}{dt} = c^2 \frac{d\rho}{dt} \quad . \tag{18}$$

Thus, we obtain Eqs. (16), (17) and (18) for a multi-component medium; these equations coincide with Eqs. (4), (5) and (7) for a medium with a homogeneous composition. The only difference is that the parameters ρ_o and c^2 appearing in Eqs. (16), (17) and (18) depend not only on the temperature but also on the concentration at the given point. Repeating the considerations of Section 8, we obtain the wave equation (10) for the acoustic pressure.

Chapter IV

SCATTERING BY INHOMOGENEITIES

The theoretical study of wave propagation in an inhomogeneous medium reduces to the integration of a wave equation with variable coefficients, a problem of great mathematical difficulty that can be solved only for a few special cases. Here we must mention the extensive investigations of Brekhovskikh [1,2,3] on wave propagation in layered media, which he carried out over the last ten or fifteen years. The problem becomes simpler in the case of a weakly inhomogeneous medium, where the parameters appearing in the wave equation deviate only slightly from their mean values. In this case, the method of small perturbations can be successfully used. In connection with the problem of wave propagation in a medium with small inhomogeneities, the method of small perturbations has been used in two different forms: the usual form and the modified form of Rytov. Thus, for example, the usual form of the method of small perturbations was used by Ellison [31], Mintzer [21,22,23], Tartarski [32] and others, in their work, while Rytov's method was used by Obukhov [12] and the author [47,48]*. Since in a certain sense the different formulations of the method of small perturbations complement each other, it seems to us expedient to study them both below and to evaluate their relative merits. For example, it is convenient to solve the scattering problem by the usual method, while it appears that it is more expedient to study amplitude and phase fluctuations in the direct wave using Rytov's method. We shall begin this chapter, which is devoted to scattering by inhomogeneities, by examining the usual form of the method of small perturbations.

10. The Method of Small Perturbations. Dropping indices, we rewrite the wave equation (10) for the acoustic pressure

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p + \nabla \log \rho \cdot \nabla p = 0 \,. \tag{19}$$

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* Recently, further work [56,57,61] using Rytov's method has appeared. In [58] the classical method of small perturbations is used.

We assume that the density and sound velocity deviate only slightly from their mean values ρ_o and c_o, i.e.

$$\rho = \rho_o + \Delta\rho, \quad c = c_o + \Delta c, \tag{20}$$

where $\Delta\rho \ll \rho_o$ and $\Delta c \ll c_o$. Confining ourselves to quantities of the first order of smallness with respect to $\Delta\rho$ and Δc, we rewrite Eq. (19) in the following form:

$$\frac{1}{c_o^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = \frac{2\Delta c}{c_o^3}\frac{\partial^2 p}{\partial t^2} - \frac{1}{\rho_o}\nabla(\Delta\rho)\cdot\nabla p. \tag{21}$$

Taking as the zeroth approximation the plane wave

$$p_o = A_o \exp[-i(\omega t - kx)] \tag{22}$$

($k = \omega/c_o$ is the wave number in the medium with the averaged characteristics), we obtain the equation

$$\frac{1}{c_o^2}\frac{\partial^2 p_1}{\partial t^2} - \nabla^2 p_1 = 2\frac{\Delta c}{c_o^3}\frac{\partial^2 p_o}{\partial t^2} - \frac{1}{\rho_o}\nabla(\Delta\rho)\cdot\nabla p_o, \tag{23}$$

for the first approximation p_1, or

$$\frac{1}{c_o^2}\frac{\partial^2 p_1}{\partial t^2} - \nabla^2 p_1 = -\left[2k^2\frac{\Delta c}{c_o} + \frac{ik}{\rho_o}\frac{\partial(\Delta\rho)}{\partial x}\right] A_o \exp[-i(\omega t - kx)]\,. \tag{24}$$

Finally, introducing the notation

$$4\pi Q = -\left[2k^2\frac{\Delta c}{c_o} + \frac{ik}{\rho_o}\frac{\partial(\Delta\rho)}{\partial x}\right] A_o \exp[-i(\omega t - kx)] \tag{25}$$

as an abbreviation, we can write Eq. (24) in the form

$$\frac{1}{c_o^2}\frac{\partial^2 p_1}{\partial t^2} - \nabla^2 p_1 = 4\pi Q . \tag{26}$$

Under the influence of the primary wave p_o each element of the inhomogeneous medium becomes a source of secondary scattered waves p_1, where Q is the density of elementary sources. The total effect of the waves scattered by the volume V is given by the solution of the inhomogeneous equation (26)

$$p_1 = \int_V \frac{Q(t - \frac{r}{c_o})}{r} \, dv , \tag{27}$$

where r is the distance from the scattering element (ξ,η,ζ) to the observation point (x,y,z), i.e.

$$r = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2} .$$

Substituting the expression (25) for Q into (27), we obtain

$$p_1 = -\frac{A_o}{4\pi}\int_V \left[2k^2 \frac{\Delta c}{c_o} + \frac{ik}{\rho_o}\frac{\partial(\Delta\rho)}{\partial\xi}\right]\frac{1}{r}\exp\left[ik(r+\xi)\right] dv \tag{28}$$

(the factor $e^{-i\omega t}$ is dropped). The scattering by fluctuations of the sound velocity is given by the first term in the square brackets, and the scattering by density fluctuations is given by the second term.

11. The Scattering Formula. The fluctuations of velocity Δc and density $\Delta\rho$ which appear in Eq. (28) are random functions of the coordinates. The transition to ordinary functions is

achieved by squaring the expression (28) and then taking the statistical average. Assuming that the fluctuations of velocity and density are caused by fluctuations of temperature (for a constant pressure p_o) we rewrite (28) as follows:

$$p_1 = - \frac{A_o}{4\pi} \int_V \left[2k^2 \frac{1}{c_o} \left(\frac{\partial c}{\partial T}\right)_{p_o} \Delta T + \frac{ik}{\rho_o} \left(\frac{\partial \rho}{\partial T}\right)_{p_o} \frac{\partial (\Delta T)}{\partial \xi} \right] \frac{1}{r} \exp\left[ik(r+\xi)\right] dv. \tag{29}$$

We shall calculate the pressure produced by the waves scattered by a cubical volume of side L which is considerably larger than the correlation distance $a (L \gg a)$ at a distance r which is considerably larger than the dimensions of the cube $(r \gg L)$. As can be seen from the diagram (Fig. 3)

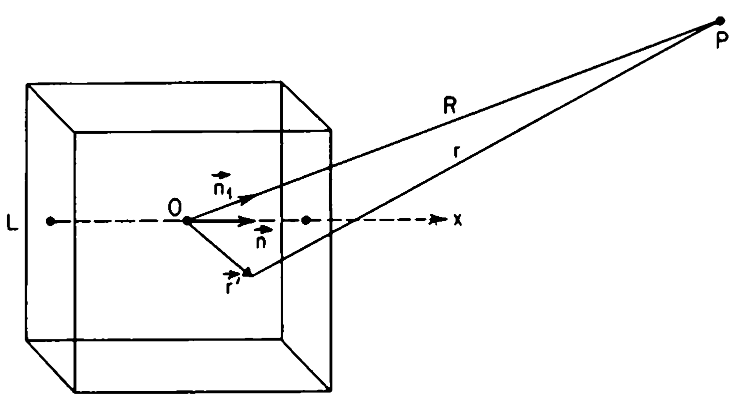

Fig. 3 The scattering configuration (schematic).

$\vec{n}_1$ is the scattering direction, $\vec{n}$ is the direction of the incident wave (coincident with the x axis), $\vec{r}'$ is the radius vector of the scattering point, r is the distance from the scattering point to the observation point, and R is the distance from the origin of coordinates to the observation point. Then we have

$$r \sim R - \vec{n}_1 \cdot \vec{r}'. \tag{30}$$

obtain

$$p_1 = -\frac{A_o}{4\pi R}\int_V \left[2k^2 \frac{1}{c_o}\left(\frac{\partial c}{\partial T}\right)_{p_o} \Delta T + \frac{ik}{\rho_o}\left(\frac{\partial \rho}{\partial T}\right)_{p_o} \frac{\partial(\Delta T)}{\partial \xi}\right] \exp\left[ikR+ik(\xi-\vec{n}_1\cdot\vec{r}\,')\right]dv. \tag{31}$$

Before proceeding, we note that Eqs. (30) and (31) can be used only in the Fraunhofer zone. In fact, Eq. (30) is obtained as a result of expanding the exact expression for r in a series of powers of r'/R and dropping all terms of the series beginning with the quadratic terms. This means that the contribution to the phase produced by the first discarded term (the quadratic term) must be small compared to π. Expanding the expression

$$r = (R^2 + r'^2 - 2\vec{R}\cdot\vec{r}\,')^{1/2}$$

in series up to squares of the ratio r'/R, we obtain

$$r \sim R\left\{1 - \frac{\vec{n}_1\cdot\vec{r}\,'}{R} + \frac{1}{2R^2}\left[r'^2 - (\vec{n}_1\cdot\vec{r}\,')^2\right]\right\} .$$

Since the largest value of $r' \sim L$, the condition that the quadratic terms be negligible can be written in the following way:

$$kL^2/R \ll \pi.$$

This condition is satisfied for the Fraunhofer zone. The minimum dimensions of the scattering volume are of the order of the correlation distance ($L \sim a$). Consequently, we obtain the least restrictive condition for R by setting $L = a$ in the preceding inequality. This gives

$$ka^2/R \ll 1.$$

Multiplying (31) by its complex conjugate, we obtain

$$|p_1|^2 = \frac{A_o^2}{16\pi^2 R^2} \int\limits_V \int\limits_V \left\{ \frac{4k^4}{c_o^2} \left(\frac{\partial c}{\partial T}\right)_{p_o}^2 \Delta T_1 \Delta T_2 + \frac{2ik^3}{\rho_o} \left(\frac{\partial \rho}{\partial T}\right)_{p_o} \cdot \frac{1}{c_o} \left(\frac{\partial c}{\partial T}\right)_{p_o} \times \right. \tag{32}$$

$$\left. \times \left[\frac{\partial}{\partial \xi_1}(\Delta T_1 \Delta T_2) - \frac{\partial}{\partial \xi_2}(\Delta T_1 \Delta T_2)\right] + \frac{k^2}{\rho_o^2}\left(\frac{\partial \rho}{\partial T}\right)_{p_o}^2 \frac{\partial^2}{\partial \xi_1 \partial \xi_2}(\Delta T_1 \Delta T_2) \right\} \times$$

$$\times \exp\left[ik\left[(\xi_1 - \xi_2) - \vec{n}_1 \cdot (\vec{r}_1' - \vec{r}_2')\right]\right] dv_1 dv_2 \, .$$

Since $\xi_1 - \xi_2 = \vec{n} \cdot (\vec{r}_1' - \vec{r}_2')$, the exponent can be transformed as follows:

$$k\left[(\xi_1 - \xi_2) - \vec{n}_1 \cdot (\vec{r}_1' - \vec{r}_2')\right] = k\left[\vec{n} \cdot (\vec{r}_1' - \vec{r}_2') - \vec{n}_1 \cdot (\vec{r}_1' - \vec{r}_2')\right] \tag{33}$$

$$= k(\vec{n} - \vec{n}_1) \cdot (\vec{r}_1' - \vec{r}_2') = \vec{K} \cdot \vec{r},$$

where we have introduced the notation

$$\vec{K} = k(\vec{n} - \vec{n}_1), \; \vec{r} = \vec{r}_1' - \vec{r}_2' \, .$$

It is easy to see that

$$K = 2k \sin \frac{\theta}{2} \, ,$$

where θ is the scattering angle; $\vec{r}$ is the distance between points inside the scattering volume. Taking the statistical average of (32) and introducing the correlation coefficient for the temperature fluctuations

$$N(\vec{r}) = \frac{\overline{\Delta T_1 \Delta T_2}}{\overline{(\Delta T)^2}},$$

we obtain

$$\overline{|p_1|^2} = \frac{A_o^2 \overline{(\Delta T)^2}}{16\pi^2 R^2} \int_V \int_V \left\{ \frac{4k^4}{c_o^2} \left(\frac{\partial c}{\partial T}\right)_{p_o}^2 N + \frac{2ik^3}{\rho_o} \left(\frac{\partial \rho}{\partial T}\right)_{p_o} \frac{1}{c_o} \left(\frac{\partial c}{\partial T}\right)_{p_o} \left(\frac{\partial N}{\partial \xi_1} - \frac{\partial N}{\partial \xi_2}\right) \right. \tag{34}$$

$$\left. + \frac{k^2}{\rho_o^2} \left(\frac{\partial \rho}{\partial T}\right)_{p_o}^2 \frac{\partial^2 N}{\partial \xi_1 \partial \xi_2} \right\} e^{i\vec{K}\cdot\vec{r}} \, dv_1 dv_2 .$$

Now introducing the relative coordinates

$$\begin{aligned} \xi &= \xi_1 - \xi_2, \\ \eta &= \eta_1 - \eta_2, \\ \zeta &= \zeta_1 - \zeta_2, \end{aligned} \tag{35}$$

and the center of mass coordinates

$$\begin{aligned} x &= \tfrac{1}{2} (\xi_1 + \xi_2), \\ y &= \tfrac{1}{2} (\eta_1 + \eta_2), \\ z &= \tfrac{1}{2} (\zeta_1 + \zeta_2), \end{aligned} \tag{36}$$

and bearing in mind that the correlation coefficient $N(\vec{r})$ depends only on the relative coordinates, we calculate the derivatives of the correlation coefficient which appear in (34)

$$\frac{\partial N}{\partial \xi_1} = \frac{\partial N}{\partial \xi}, \qquad \frac{\partial N}{\partial \xi_2} = - \frac{\partial N}{\partial \xi}, \qquad \frac{\partial^2 N}{\partial \xi_1 \partial \xi_2} = - \frac{\partial^2 N}{\partial \xi^2} .$$

Then Eq. (34) takes the form

$$\overline{|p_1|^2} = \frac{A_o^2\overline{(\Delta T)^2}}{16\pi^2R^2} \int_V\int_V \left(\frac{4k^4}{c_o^2} \left(\frac{\partial c}{\partial T}\right)^2_{p_o} N + \frac{2ik^3}{\rho_o} \left(\frac{\partial \rho}{\partial T}\right)_{p_o} \frac{1}{c_o} \left(\frac{\partial c}{\partial T}\right)_{p_o} 2\,\frac{\partial N}{\partial \xi}\right. \tag{37}$$

$$\left. - \frac{k^2}{\rho_o^2} \left(\frac{\partial \rho}{\partial T}\right)^2_{p_o} \frac{\partial^2 N}{\partial \xi^2}\right) e^{i\vec{K}\cdot\vec{r}}\, dv dv_o .$$

The integrand does not depend on the center of mass coordinates. Carrying out the integration with respect to these coordinates, we simply obtain the factor V. The integral with respect to the relative coordinates reduces to three integrals, and integration by parts is used to evaluate the integrals containing derivatives of the correlation coefficient $N(\vec{r})$. In fact, noting that $\vec{K}\cdot\vec{r} = K_\xi \xi + K_\eta \eta + K_\zeta \zeta$, we obtain

$$\iiint_{-L/2}^{+L/2} e^{i\vec{K}\cdot\vec{r}} \frac{\partial N}{\partial \xi}\, d\xi d\eta d\zeta = \iint_{-L/2}^{+L/2} \left[e^{i\vec{K}\cdot\vec{r}}\, N\right]_{-L/2}^{+L/2} d\eta d\zeta - \iiint_{-L/2}^{+L/2} iK_\xi e^{i\vec{K}\cdot\vec{r}}\, N d\xi d\eta d\zeta$$

$$= - iK_\xi \int_V e^{i\vec{K}\cdot\vec{r}}\, N dv ,$$

$$\iiint_{-L/2}^{+L/2} e^{i\vec{K}\cdot\vec{r}} \frac{\partial^2 N}{\partial \xi^2}\, d\xi d\eta d\zeta = \iint_{-L/2}^{+L/2} \left[e^{i\vec{K}\cdot\vec{r}} \frac{\partial N}{\partial \xi}\right]_{-L/2}^{+L/2} d\eta d\zeta - \iiint_{-L/2}^{+L/2} iK_\xi e^{i\vec{K}\cdot\vec{r}} \frac{\partial N}{\partial \xi}\, d\xi d\eta d\zeta$$

$$= - \iint_{-L/2}^{+L/2} \left[iK_\xi e^{i\vec{K}\cdot\vec{r}}\, N\right]_{-L/2}^{+L/2} d\eta d\zeta - \iiint_{-L/2}^{+L/2} K_\xi^2\, e^{i\vec{K}\cdot\vec{r}}\, N d\xi d\eta d\zeta = - K_\xi^2 \int_V e^{i\vec{K}\cdot\vec{r}}\, N dv,$$

since the correlation coefficient and its derivative vanish on the faces $\xi = \pm \frac{L}{2}$ of the cube, the dimensions of which are large compared to the correlation distance. Because of these

relations, Eq. (37) becomes

$$\overline{|p_1|^2} = V \frac{A_o^2 \overline{(\Delta T)^2}}{16\pi^2 R^2} \left[\frac{2k^2}{c_o} \left(\frac{\partial c}{\partial T}\right)_{p_o} + \frac{K_\xi k}{\rho_o} \left(\frac{\partial \rho}{\partial T}\right)_{p_o}\right]^2 \int_V N(\vec{r})\, e^{i\vec{K}\cdot\vec{r}}\, dv , \tag{38}$$

where

$$K_\xi = \vec{K}\cdot\vec{n} = k(\vec{n} - \vec{n}_1)\cdot\vec{n} = k(1 - \vec{n}_1\cdot\vec{n}) = k(1 - \cos\theta) = 2k \sin^2\frac{\theta}{2} . \tag{39}$$

Using (39) to replace K_ξ in (38), we finally obtain the following formula:

$$\overline{|p_1|^2} = V \frac{A_o^2 \overline{(\Delta T)^2}\, k^4}{4\pi^2 R^2} \left[\frac{1}{c_o} \left(\frac{\partial c}{\partial T}\right)_{p_o} + \frac{1}{\rho_o} \left(\frac{\partial \rho}{\partial T}\right)_{p_o} \sin^2\frac{\theta}{2}\right]^2 \int_V N(\vec{r})\, e^{i\vec{K}\cdot\vec{r}}\, dv. \tag{40}$$

Eq. (40) lets us compare the scattering by velocity fluctuations with the scattering by density fluctuations: these correspond to the first and second terms in the square brackets preceding the integral, respectively.

In the case of a gaseous medium at the constant pressure p_o, the sound velocity fluctuations are given in terms of the density fluctuations, as can be seen from the formula

$$c^2 = \frac{\gamma p_o}{\rho} ,$$

for the velocity of sound, so that

$$\frac{2}{c_o} \left(\frac{\partial c}{\partial T}\right)_{p_o} = - \frac{1}{\rho_o} \left(\frac{\partial \rho}{\partial T}\right)_{p_o} . \tag{41}$$

Consequently, in a gas the scattering by velocity fluctuations and the scattering by density fluctuations agree in order of magnitude. In liquids Eq. (41) is no longer valid and the left hand side predominates. Thus, for example, according to the data of Beranek [34],

$$\frac{1}{c_o}\left(\frac{\partial c}{\partial T}\right)_{p_o} \sim 2 \times 10^{-3}, \quad \frac{1}{\rho_o}\left(\frac{\partial \rho}{\partial T}\right)_{p_o} \sim 2.6 \times 10^{-4},$$

for sea water at 15° C and with a salinity of 36%, i.e., the velocity fluctuations are an order of magnitude larger than the density fluctuations. For this reason, bearing in mind subsequent applications of the scattering formula in hydroacoustics, we shall neglect density fluctuations in comparison with sound velocity fluctuations. This cannot usually be done in atmospheric acoustics (this matter is discussed in more detail below). For water Eq. (40) simplifies to

$$\overline{|p_1|^2} = V \frac{A_o^2 \overline{(\Delta T)^2} k^4}{4\pi^2 R^2} \frac{1}{c_o^2}\left(\frac{\partial c}{\partial T}\right)_{p_o}^2 \int_V N(\vec{r}) e^{i\vec{K}\cdot\vec{r}} dv. \tag{42}$$

Note that neglecting density fluctuations is equivalent to neglecting the additional term $\nabla \log \rho \cdot \nabla p$ in the original wave equation (19).

The form of Eq. (42) can be abbreviated a bit by introducing the refractive index n instead of the velocity c ($n = c_o/c$). Denoting by μ the deviation of the refractive index from its mean value (equal to unity), we find from $n = c_o/c$ that

$$\begin{aligned} \mu &= \Delta n = -\frac{1}{c_o}\left(\frac{\partial c}{\partial T}\right)_{p_o} \Delta T, \\ \overline{\mu^2} &= \frac{1}{c_o^2}\left(\frac{\partial c}{\partial T}\right)^2 \overline{(\Delta T)^2}. \end{aligned} \tag{43}$$

Eq. (42) can then be rewritten in the following way:

$$\overline{|p_1|^2} = V \frac{A_o^2 \, k^4 \, \overline{\mu^2}}{4\pi^2 \, R^2} \int_V N(\vec{r}) \, e^{i\vec{K}\cdot\vec{r}} \, dv. \tag{44}$$

This is the form in which it is encountered in Pekeris' paper [35].

If the statistical properties of the medium are isotropic, then the correlation coefficient depends only on the modulus of $\vec{r}$. In this case it is expedient to introduce a spherical coordinate system, the polar axis of which coincides with the vector $\vec{K}$, and to carry out the angular integrations, i.e.

$$\int_0^\infty \int_0^\pi \int_0^{2\pi} N(r) \, e^{iKr \cos\alpha} \, r^2 dr \sin \alpha d\alpha d\phi = \frac{4\pi}{K} \int_0^\infty N(r) \sin(Kr) \, r dr.$$

(The upper limit in the integral with respect to r can be set equal to ∞, since the dimensions of the cube are large compared to the correlation distance.) Eq. (44) becomes

$$\overline{|p_1|^2} = V \frac{A_o^2 \, k^4 \, \overline{\mu^2}}{\pi R^2 K} \int_0^\infty N(r) \sin(Kr) \, r dr. \tag{45}$$

The integral in this formula can be evaluated by specifying the form of the function N(r). For example, setting

$$N(r) = e^{-r/a} , \tag{46}$$

we obtain the tabulated integral

$$\int_0^\infty e^{-r/a} \sin(Kr) \, r dr = \frac{2Ka^3}{\left(1 + 4k^2 a^2 \sin^2 \frac{\theta}{2}\right)^2} .$$

In this case the scattering formula can be written in the following way:

$$\overline{|p_1|^2} = V \frac{2A_o^2 k^4 a^3 \overline{\mu^2}}{\pi R^2} \frac{1}{\left(1 + 4k^2 a^2 \sin^2 \frac{\theta}{2}\right)^2}. \tag{47}$$

This formula can also be found in the paper of Pekeris cited above. Setting

$$N(r) = e^{-r^2/a^2}, \tag{48}$$

we again obtain a tabulated integral and the scattering formula becomes

$$\overline{|p_1|^2} = V \frac{k^4 a^3 \overline{\mu^2} A_o^2}{4\sqrt{\pi} R^2} \exp\left[-k^2 a^2 \sin^2 \frac{\theta}{2}\right]. \tag{49}$$

From Eqs. (47) and (49) we can draw the following conclusions, which will be important later:

1. "Small-scale" fluctuations ($ka \ll 1$) are the cause of isotropic scattering.

2. In the case of "large-scale" fluctuations ($ka \gg 1$), the scattering has a sharply directed character, and most of the scattered power is concentrated within the small solid angle $\theta \sim 1/ka$.

In atmospheric acoustics it is usually necessary to take into account density fluctuations as well as velocity fluctuations. As can be seen from Eq. (40), the scattering by density fluctuations corresponds to the term

$$\frac{1}{\rho_o} \left(\frac{\partial \rho}{\partial T}\right)_{p_o} \sin^2 \frac{\theta}{2},$$

which has a maximum in the direction $\theta = \pi$ (reflection). However, in the case of large-scale inhomogeneities this term is suppressed by the factor

$$\int_V N(\vec{r})\, e^{i\vec{K}\cdot\vec{r}}\, dv,$$

which has a strong minimum in the same direction. In other words, in atmospheric acoustics we can neglect density fluctuations as compared with velocity fluctuations, for large-scale inhomogeneities; in hydroacoustics we can do so for any relation between the wave length and the size of the inhomogeneities Moreover, these conclusions remain valid in the case where the velocity and density fluctuations are caused by concentration fluctuations. In this case the formulas will contain derivatives with respect to concentration instead of derivatives with respect to temperature. For sea water [30] we have

$$\frac{1}{c_o}\left(\frac{\partial c}{\partial C}\right)_{T,p_o} \sim 0.8\,, \qquad \frac{1}{\rho_o}\left(\frac{\partial \rho}{\partial C}\right)_{T,p_o} \sim 0.15 - 0.20,$$

i.e., the velocity fluctuations are 4 to 5 times larger than the density fluctuations.

We can obtain similar scattering formulas for electromagnetic waves. They differ from the formulas obtained above using the scalar wave equation by the presence of the factor $\sin^2 X$, where X is the angle between the scattering direction and the direction of the electric vector in the incident wave. This factor takes into account the nature of the polarization of the incident wave.

In their theoretical investigation of the scattering of radio waves in the troposphere Booker and Gordon [36] assume that the correlation coefficient of the dielectric constant has the form $e^{-r/a}$; as a result, they obtain the scattering formula (47) which is valid in this special case. Fejer [41] used a correlation coefficient of the form e^{-r^2/a^2} and obtained Eq. (49). Staras [37] applied the method of small perturbations to Maxwell's equations without any special assumptions about the correlation coefficient and obtained the general scattering formula (44).

12. The Scattering Coefficient. The attenuation of energy flow in a wave which passes through a layer of thickness L is given by the formula $\Delta I = \alpha IL$, from which we obtain the relation

$$\alpha = \frac{\Delta I}{IL} \tag{50}$$

for the scattering coefficient. The energy flow I through the boundary of the cube is pro-

$$I = \beta A_o^2 L^2, \tag{51}$$

where β is a constant of proportionality. The attenuation ΔI of the flow in going a distance L is equal to the total energy scattered by the cube, i.e.

$$\Delta I = \beta \oint \overline{|p_1|^2}\, ds. \tag{52}$$

Here the integral is taken over the surface of a sphere of radius R. Confining ourselves to the case of a statistically isotropic medium and using Eqs. (45) and (50), we obtain

$$\alpha = \frac{k^4\, \overline{\mu^2}}{\pi} \int d\Omega \int_0^\infty N(r) \frac{\sin(Kr)}{K}\, rdr, \tag{53}$$

where $d\Omega$ is the element of solid angle. Recalling that $K = 2k \sin \frac{\theta}{2}$ and changing the order of integration, we rewrite Eq. (53) in the following form:

$$\alpha = 2\, \overline{\mu^2}\, k^3 \int_0^\infty N(r)\, rdr \int_0^\pi \sin(2kr \sin \tfrac{\theta}{2}) \cos \tfrac{\theta}{2}\, d\theta. \tag{54}$$

Integrating with respect to θ, we finally get the formula

$$\alpha = 2\, \overline{\mu^2}\, k^2 \int_0^\infty \left[1 - \cos(2kr)\right] N(r) dr. \tag{55}$$

We now calculate the scattering coefficients for the different correlation coefficients. Setting $N = e^{-r/a}$, we find

$$\alpha = \frac{8\,\overline{\mu^2}\,k^4a^3}{1 + 4k^2a^2}\,. \tag{56}$$

Setting $N = e^{-r^2/a^2}$, we obtain

$$\alpha = \sqrt{\pi}\,\overline{\mu^2}\,k^2a(1 - e^{-k^2a^2}). \tag{57}$$

In the case of small-scale inhomogeneities ($ka \ll 1$) these formulas become

$$\alpha = 8\,\overline{\mu^2}\,k^4a^3, \qquad \alpha = \sqrt{\pi}\,\overline{\mu^2}\,k^4a^3. \tag{58}$$

Thus, as was to be expected, for small-scale fluctuations we get the Rayleigh dependence: the scattering coefficient is proportional to the fourth power of the frequency. For large-scale fluctuations ($ka \gg 1$), we have

$$\alpha = 2\,\overline{\mu^2}\,k^2a, \qquad \alpha = \sqrt{\pi}\,\overline{\mu^2}\,k^2a, \tag{59}$$

i.e., in the case of large-scale inhomogeneities the scattering coefficient increases as the square of the frequency, like the coefficient of viscous absorption.

13. Applicability of the Scattering Formula. The condition for the applicability of the scattering formula can be obtained from the requirement that the scattered energy be small compared to the incident energy, i.e. from the condition $\Delta I \ll I$, which by (50) gives

$$\alpha L \ll 1. \tag{61}$$

Considering the case of large-scale inhomogeneities, we use (59) to replace α in (61) and obtain

$$\overline{\mu^2}\,k^2aL \ll 1. \tag{62}$$

In particular, if the scattering volume has dimensions of the order of the scale of the inhomogeneities ($L \sim a$), then the condition (62) for the applicability of the scattering formula can be written as

$$\overline{\mu^2}\, k^2 a^2 \ll 1 \quad \text{or} \quad \sqrt{\overline{\mu^2}}\; ka \ll 1. \tag{63}$$

The inequality (63) also follows from an analysis of the suitability of an approximate solution of the problem of scattering by a sphere of radius a, whose properties deviate only slightly from the properties of the surrounding medium [35]. In our case, of course, the role of the radius is played by the characteristic scale of the inhomogeneities (the correlation distance).

It follows from the condition (62) that the dimensions of the scattering volume and the scale of the inhomogeneities must not be too large, and the frequencies must not be too high. In particular, this means that it is impossible to make the transition to geometrical acoustics ($k \to \infty$) in the scattering formula. In this case the formulas of the preceding section would give infinite values.

14. Attenuation of an Acoustic Bundle Due to Scattering. The results of Section 12 pertain to a plane-parallel bundle. If the bundle diverges, as is often the case, then in calculating the scattering coefficient it is necessary to take into account that part of the energy which is scattered out of the bundle. The energy scattered inside the bundle does not cause it to be attenuated but rather causes another effect, namely amplitude and phase fluctuations in the direct bundle (see Chapter 5).

If an axially symmetric bundle has a divergence angle $2\theta_o$, then in calculating the scattered energy and the scattering coefficient using Eq. (54) it is necessary to integrate with respect to θ from the limits θ_o to π. Then we obtain

$$\alpha = 2\, \overline{\mu^2}\, k^2 \int_0^\infty N(r) \left[\cos\left(2kr \sin \frac{\theta_o}{2}\right) - \cos(2kr)\right] dr. \tag{64}$$

Setting $N = e^{-r/a}$ and carrying out the integration in (64), we obtain

$$\alpha = 8\,\overline{\mu^2}\,k^4a^3 \;\frac{1 - \sin^2\dfrac{\theta_o}{2}}{(1 + 4k^2a^2)\left(1 + 4k^2a^2\sin^2\dfrac{\theta_o}{2}\right)}\;. \tag{65}$$

If $N = e^{-r^2/a^2}$, then (64) gives

$$\alpha = \sqrt{\pi}\,\overline{\mu^2}\,k^2a \left(\exp\left[-k^2a^2\sin^2\frac{\theta_o}{2}\right] - e^{-k^2a^2}\right)\;. \tag{66}$$

In the case of large-scale inhomogeneities most of the scattered energy is concentrated inside the angle $\theta = 1/ka$. This angle can be larger than or smaller than the divergence angle θ_o of the bundle. If $\theta < \theta_o$, most of the energy is scattered inside the bundle. In the other limiting case ($\theta \gg \theta_o$) most of the scattered energy leaves the boundaries of the bundle. In the latter case in calculating the scattering coefficient we can use the formulas derived for a plane wave.

Liebermann [4] gives an incorrect formula for a plane-parallel bundle, which differs from Eq. (56) by the factor $4k^2a^2$; as a result Liebermann gets a value of the scattering coefficient which is three orders of magnitude too small! In fact, using his experimental data ($\overline{\mu^2} = 5 \times 10^{-9}$, $a = 60$ cm) Liebermann gets $\alpha \sim 10^{-10}$ cm^{-1} for a frequency $\nu = 10$ kilocycles. The calculation based on Eq. (56) gives $\alpha = 10^{-7}$ cm^{-1} = 0.04 db/km, which is approximately a twentieth of the observed attenuation in sea water at the same frequency. Since the scattering and absorption coefficients depend in the same way on frequency (proportional to the square of the frequency) the ratio just obtained between the scattering and absorption coefficients remains the same (in order of magnitude) at other frequencies too.

Chapter V

FLUCTUATIONS

When a wave propagates in a medium with random inhomogeneities, fluctuations of the characteristics of the wave field due to the superposition of the scattered waves and the primary wave are observed. There must be a dependence between the fluctuations of the characteristics of the wave field and the fluctuations of refractive index. The theoretical problem consists in finding this dependence, which can then be used to draw conclusions about the statistical properties of the wave field, given the statistical properties of the medium. The inverse problem is usually not unique and cannot be solved without additional assumptions. However, if we make reasonable assumptions about the form of the correlation coefficient of the refractive index, for example, then by measuring the field fluctuations we can determine the mean value of the refractive index fluctuations. Thus, the study of wave fluctuations, which is interesting in its own right, opens new possibilities for the study of the properties of the medium which the wave traverses.

We begin our study of the statistical properties of the wave field by examining amplitude and phase fluctuations. In order to do so, we shall derive the basic formulas for the amplitude and phase fluctuations by two different methods; this will allow us to compare the merits of the two methods.

15. The Method of Small Perturbations. In order to be explicit, we shall assume that the random inhomogeneities occur only in the right half-space ($x > 0$) and that the left half-space ($x < 0$) contains no random inhomogeneities. A plane wave

$$p_o = A_o e^{i\phi_o} = A_o e^{-i(\omega t-kx)}.$$

advances from the homogeneous to the inhomogeneous medium. A receiver is located in the inhomogeneous medium at the point with coordinates x,y,z, and the waves scattered by the inhomogeneities as well as the wave p_o are incident on the receiver. We shall describe their

total effect by the wave function

$$p_1 = A_1 \, e^{i\phi_1},$$

which can easily be found to a first approximation by using the method of small perturbations, if we take p_o as the zeroth approximation. Neglecting density fluctuations as compared with velocity fluctuations and using (28), we obtain

$$p_1 = \frac{k^2 A_o}{2\pi} \int_V \frac{e^{ik(r+\xi)}}{r} \mu(\xi,\eta,\zeta) \, dv , \tag{67}$$

where r is the distance from the scattering element dv with coordinates ξ,η,ζ to the observation point (x,y,z). The integration in (67) is to be taken over that part of space from which scattered waves arrive at the observation point. If we denote the result of superimposing the primary and scattered waves by

$$p = A \, e^{i\phi},$$

then we can write the following equation:

$$A \, e^{i\phi} = A_o \, e^{i\phi_o} + A_1 \, e^{i\phi_1} .$$

Dividing both sides by $A_o e^{i\phi_o}$, we obtain

$$\frac{A}{A_o} \, e^{i(\phi-\phi_o)} = 1 + \frac{A_1}{A_o} \, e^{i(\phi_1-\phi_o)}, \tag{68}$$

where by (67) the last term is given by the formula

$$\frac{A_1}{A_o} \, e^{i(\phi_1-\phi_o)} = \frac{k^2}{2\pi} \int_V \frac{1}{r} \, e^{ik[r-(x-\xi)]} \mu(\xi,\eta,\zeta) \, dv . \tag{69}$$

Introducing the notation

$$\frac{A_1}{A_o} e^{i(\phi_1 - \phi_o)} = X + iY, \tag{69a}$$

we find

$$X = \frac{k^2}{2\pi} \int \frac{1}{r} \cos k\left[r - (x - \xi)\right] \mu(\xi,\eta,\zeta) \, dv, \tag{70}$$

$$Y = \frac{k^2}{2\pi} \int \frac{1}{r} \sin k\left[r - (x - \xi)\right] \mu(\xi,\eta,\zeta) \, dv . \tag{70a}$$

Since the amplitude of the scattered waves A_1 is small compared to the amplitude of the primary wave A_o, the amplitude $A = A_o + \Delta A$ and phase $\phi = \phi_o + \Delta\phi$ of the resulting wave differ only slightly from the amplitude A_o and phase ϕ_o of the primary wave, i.e.

$$\frac{\Delta A}{A_o} \ll 1, \qquad \Delta\phi \ll 1.$$

Expanding the left hand side of Eq. (68) in powers of the small fluctuations and keeping only leading terms, we obtain

$$\frac{\Delta A}{A_o} + i\Delta\phi = X + iY.$$

Equating real and imaginary parts separately and using (70) and (70a), we obtain the following formulas

$$\Delta\phi = \frac{k^2}{2\pi} \int \frac{1}{r} \sin k\left[r - (x - \xi)\right] \mu(\xi,\eta,\zeta) \, dv, \tag{71}$$

$$\frac{\Delta A}{A_o} = \frac{k^2}{2\pi} \int \frac{1}{r} \cos k\left[r - (x - \xi)\right] \mu(\xi,\eta,\zeta) \, dv \tag{72}$$

for the phase fluctuation $\Delta\phi$ and the amplitude fluctuation ΔA. In these formulas the argument $[r-(x-\xi)]$ gives the difference in path length to the observation point (x,y,z) traversed by the primary wave and by the wave scattered by the element (ξ,η,ζ). We note that this difference becomes zero if the scattering element lies on a straight line through the observation point and parallel to the x-axis, as is to be expected in this case.

The expressions just obtained for the amplitude and phase fluctuations are restricted by the requirement that $\Delta\phi \ll 1$ and $\Delta A/A_o \ll 1$, and consequently by the requirement that the distances be small, since the fluctuations increase with distance. Therefore, it is more expedient to use the method of small perturbations in the form given by Rytov [11] in connection with the theory of diffraction of light by an ultrasonic lattice. As we shall see below, in this form the method of small perturbations is not subject to such a stringent restriction; this is essential for comparing theory with experiment, since experimentally one often observes large amplitude and phase fluctuations.

16. Rytov's Method. The acoustic pressure satisfies the wave equation

$$\frac{(1+\mu)^2}{c_o^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = 0, \tag{74}$$

where μ is the deviation of the refractive index of the medium from its mean value of unity and $|\mu| \ll 1$. As before, we write the plane wave in the left half-space in the form

$$p_o = A_o\, e^{-i(\omega t - kx)}.$$

We are looking for a solution of the wave equation in the right half-space in the form

$$p = A(\vec{r})\, e^{-i[\omega t - S(\vec{r})]}, \tag{75}$$

where $A(\vec{r})$ and $S(\vec{r})$ are unknown functions. The gist of the method consists in replacing the wave function p by another function ψ which is connected with the first function by the relation

$$p = A_o e^{-i[\omega t - \psi(\vec{r})]}. \tag{76}$$

It follows by comparing (75) and (76) that the function $\psi(\vec{r})$ is given by the following relation:

$$\psi(\vec{r}) = S(\vec{r}) - i \log \frac{A(\vec{r})}{A_o} . \tag{77}$$

As is apparent, the real and imaginary parts of the function $\psi(\vec{r})$ just introduced determine the phase and the logarithm of the ratio of amplitudes (sound intensities), respectively; the latter are the fluctuations in which we are primarily interested.

Using (76) to replace the function p by ψ in Eq. (74), we obtain the following equation for the function ψ:

$$(\nabla\psi)^2 - i\nabla^2\psi = k^2n^2. \tag{78}$$

The zeroth approximation ψ_o satisfies the wave equation for the homogeneous medium, i.e.

$$(\nabla\psi_o)^2 - i\nabla^2\psi_o = k^2. \tag{79}$$

Subtracting (79) from (78), we obtain the equation

$$2(\nabla\psi_o \cdot \nabla\psi') - i\nabla^2\psi' = 2\mu k^2 + \left[\mu^2 k^2 - (\nabla\psi')^2\right] \tag{80}$$

for the function $\psi' = \psi - \psi_o$. Using the method of small perturbations and assuming that $\nabla\psi'$ (more accurately, the dimensionless quantity $\frac{1}{k}\nabla\psi'$) is of order μ, we drop the terms contained in the square brackets, since they are small quantities of the second order with respect to μ. Then for ψ' we obtain the linear equation

$$2(\nabla\psi_o \cdot \nabla\psi') - i\nabla^2\psi' = 2\mu k^2. \tag{81}$$

This equation was obtained under the assumption that $\frac{1}{k} \nabla\psi' \sim \mu$ or

$$\frac{1}{k} |\nabla\psi'| \ll 1. \tag{82}$$

This last condition means that the phase change and the relative change of amplitude <u>per wavelength</u> must be small. The relation (82) imposes no restrictions on the <u>total</u> change of these quantities. Setting $\psi_o = kx$ in (81), we obtain

$$2k \frac{\partial\psi'}{\partial x} - i\nabla^2\psi' = 2\mu k^2. \tag{33}$$

Introducing the new function W defined by

$$\psi' = e^{-ikx} W, \tag{84}$$

we find for it the equation

$$\nabla^2 W + k^2 W = f(x,y,z), \tag{85}$$

where

$$f(x,y,z) = i2\mu k^2 e^{ikx} . \tag{86}$$

The solution of the inhomogeneous equation (85) has the form

$$W = -\frac{1}{4\pi} \int \frac{1}{r} e^{\pm ikr} f(\xi,\eta,\zeta)\, dv, \tag{87}$$

where

$$r = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2} .$$

We must take the plus sign in the exponent so that the formulas of this section will reduce in the case of small amplitude and phase fluctuations to the formulas of the preceding section, which were obtained by using the usual form of the method of small perturbations.

Using (86) to replace f, we obtain

$$W = -\frac{ik^2}{2\pi}\int \frac{1}{r}\, e^{ik(r+\xi)}\, \mu(\xi,\eta,\zeta)\, dv. \tag{88}$$

Using (84), we now return to the function ψ'

$$\psi' = -\frac{ik^2}{2\pi}\int \frac{1}{r}\, e^{ik\left[r-(x-\xi)\right]}\, \mu(\xi,\eta,\zeta)\, dv. \tag{89}$$

Using (77) to separate real and imaginary parts, we obtain the following formulas

$$S' = \frac{k^2}{2\pi}\int \frac{1}{r}\sin k\left[r - (x - \xi)\right]\mu dv, \tag{90}$$

$$\log \frac{A(\vec{r})}{A_o} = \frac{k^2}{2\pi}\int \frac{1}{r}\cos k\left[r - (x - \xi)\right]\mu dv \tag{91}$$

for the amplitude and phase fluctuations. In the case of small fluctuations they reduce to Eqs. (71) and (72) of the preceding section, respectively, which were obtained by using the usual form of the method of small perturbations.

We now discuss the limits of applicability of Rytov's method. These limits are a consequence of the inequality (82)

$$\frac{1}{k}\,|\nabla\psi'| \ll 1. \tag{92}$$

This inequality implies the smallness of the relative amplitude change over a wavelength

$$\frac{1}{k}\left|\nabla \log \frac{A(\vec{r})}{A_o}\right| \ll 1 \tag{93}$$

and the smallness of the change of the phase fluctuation over a wavelength

$$\frac{1}{k}\,|\nabla S'| \leq 1. \tag{94}$$

The inequality (93) is met if the amount of scattering in going a wavelength is small. The inequality (94) implies that the angle of inclination of the ray to the initial direction is small. In fact, since

$$S = S_0 + S' = kx + S'(x,y,z), \tag{95}$$

we have

$$\frac{\partial S}{\partial x} = k + \frac{\partial S'}{\partial x}, \qquad \frac{\partial S}{\partial y} = \frac{\partial S'}{\partial y}, \qquad \frac{\partial S}{\partial z} = \frac{\partial S'}{\partial z}. \tag{96}$$

By (94) the relations

$$\left|\frac{\partial S}{\partial x}\right| \sim k, \qquad \left|\frac{\partial S}{\partial y}\right| \ll k, \qquad \left|\frac{\partial S}{\partial z}\right| \ll k, \tag{97}$$

are valid, i.e., the transverse components of the gradient of phase are small compared to the longitudinal component. Therefore, the angle of inclination of the ray with the initial direction is small.

The condition that the amount of scattering in going a wavelength be small is always fulfilled in a weakly inhomogeneous medium ($|\mu| \ll 1$). Since large-scale inhomogeneities produce sharply directed scattering, in this case the condition that the angle of inclination of the ray be small is also met. Small-scale inhomogeneities produce isotropic scattering, so that in this case the condition that the angle of inclination of the ray be small is met only when the amplitude of the scattered waves is small compared to the amplitude of the incident wave.

17. Comparison of the Methods. After studying both methods and finding their limits of applicability, it is natural to compare their relative merits. The usual form of the method of small perturbations is subject to a distance limitation. In the case of large-scale inhomogeneities, Rytov's method is not subject to a distance limitation, and is therefore more gene-

ral. In the case of small-scale inhomogeneities the two methods are equivalent. Since large-scale inhomogeneities are observed under actual conditions, we shall give preference to Rytov's method.

18. The Fresnel Approximation. In the case of large-scale inhomogeneities ($ka \gg 1$) we can neglect wave reflection and limit the region of integration in Eqs. (90) and (91) to the layer which lies in front of the receiver and is contained between the planes $\xi = 0$ and $\xi = x$. An appreciable effect will be produced only by those inhomogeneities which are concentrated within a cone with its vertex at the receiving point and with an aperture angle of the order of $1/ka$. Inside this cone the formula

$$r = \sqrt{(x-\xi)^2 + \rho^2} \, ,$$

where

$$\rho^2 = (y - \eta)^2 + (z - \zeta)^2,$$

can be replaced by the approximation

$$r \sim (x - \xi) + \frac{1}{2} \frac{\rho^2}{x - \xi} \, . \tag{98}$$

Using (98) to substitute for $r-(x-\xi)$ in (90) and (91) and replacing $1/r$ by $1/(x-\xi)$, we obtain

$$S' = \frac{k^2}{2\pi} \int_0^x \int_{-\infty}^{+\infty} \int \frac{\sin\left[k\rho^2/2(x-\xi)\right]}{x - \xi} \mu(\xi,\eta,\zeta) \, d\xi d\eta d\xi, \tag{99}$$

$$\log \frac{A(\vec{r})}{A_0} = \frac{k^2}{2\pi} \int_0^x \int_{-\infty}^{+\infty} \int \frac{\cos\left[k\rho^2/2(x-\xi)\right]}{x - \xi} \mu(\xi,\eta,\zeta) \, d\xi d\eta d\zeta. \tag{100}$$

Eqs. (99) and (100) correspond to the Fresnel approximation in diffraction theory. In Appendix I we give a more rigorous justification of these equations, using the Sommerfeld integral.

We introduce the following abbreviations in the integrands of (99) and (100):

$$\Phi_1 \left(\frac{x - \xi}{k}, \rho \right) \equiv \frac{k}{2\pi(x - \xi)} \sin \frac{k\rho^2}{2(x - \xi)},$$

$$\Phi_2 \left(\frac{x - \xi}{k}, \rho \right) \equiv \frac{k}{2\pi(x - \xi)} \cos \frac{k\rho^2}{2(x - \xi)}. \tag{101}$$

Moreover, we drop the prime in writing the phase fluctuation S' and denote the amplitude fluctuation by $B = \log A/A_o$. Then Eqs. (99) and (100) can be rewritten as

$$S(x,y,z) = k \int_0^x \int_{-\infty}^{+\infty} \int \Phi_1 \left(\frac{x - \xi}{k}, \rho \right) \mu(\xi,\eta,\zeta) \, d\xi d\eta d\zeta, \tag{102}$$

$$B(x,y,z) = k \int_0^x \int_{-\infty}^{+\infty} \int \Phi_2 \left(\frac{x - \xi}{k}, \rho \right) \mu(\xi,\eta,\zeta) \, d\xi d\eta d\zeta. \tag{103}$$

Eqs. (102) and (103) were obtained by Obukhov [12]. We now introduce the dimensionless variables

$$x' = kx, \; y' = ky, \; z' = kz, \; \xi' = k\xi, \; \eta' = k\eta, \; \zeta' = k\zeta, \; \rho' = k\rho.$$

Then finally Eqs. (102) and (103) can be written in the following form:

$$S(x',y',z') = \int_0^{x'} \int_{-\infty}^{+\infty} \int \Phi_1(x' - \xi',\rho') \, \mu(\xi',\eta',\zeta') \, d\xi' d\eta' d\zeta', \tag{104}$$

$$B(x',y',z') = \int_0^{x'} \int_{-\infty}^{+\infty} \int \Phi_2(x' - \xi',\rho') \, \mu(\xi',\eta',\zeta') \, d\xi' d\eta' d\zeta'. \tag{105}$$

19. Amplitude and Phase Fluctuations. Turning now to the statistical aspects of the problem, we undertake the calculation of the mean square amplitude and phase fluctuations. We assume that the receiver is located at the point $(L,0,0)$ or $(L',0,0)$, where $L' = kL$. Then Eqs. (104) and (105) can be rewritten as

$$S(L',0,0) = \int_0^{L'}\int_{-\infty}^{+\infty}\int \Phi_1(L'-\xi',\rho')\ \mu(\xi',\eta',\zeta')d\xi'd\eta'd\zeta', \tag{106}$$

$$B(L',0,0) = \int_0^{L'}\int_{-\infty}^{+\infty}\int \Phi_2(L'-\xi',\rho')\ \mu(\xi',\eta',\zeta')d\xi'd\eta'd\zeta', \tag{107}$$

where $\rho' = \sqrt{\eta'^2 + \zeta'^2}$. Squaring (106) and (107) and averaging, we obtain

$$\overline{S^2} = \overline{\mu^2}\int_0^{L'}\int_0^{L'}\int\int_{-\infty}^{+\infty}\int\int \Phi_1(L'-\xi_1',\rho_1')\Phi_1(L'-\xi_2',\rho_2')N(r')d\xi_1'd\xi_2'd\eta_1'd\eta_2'd\zeta_1'd\zeta_2', \tag{108}$$

$$\overline{B^2} = \overline{\mu^2}\int_0^{L'}\int_0^{L'}\int\int_{-\infty}^{+\infty}\int\int \Phi_2(L'-\xi_1',\rho_1')\Phi_2(L'-\xi_2',\rho_2')N(r')d\xi_1'd\xi_2'd\eta_1'd\eta_2'd\zeta_1'd\zeta_2', \tag{109}$$

where $N(r')$ is the correlation coefficient of the refractive index and

$$r' = \sqrt{(\xi_1'-\xi_2')^2 + (\eta_1'-\eta_2')^2 + (\zeta_1'-\zeta_2')^2}.$$

We shall consider only statistically isotropic media. In this case the correlation coefficient N depends only on the modulus of $\vec{r}$. Introducing relative coordinates

$$\eta = \eta_1' - \eta_2' \ , \ \zeta = \zeta_1' - \zeta_2' \tag{110}$$

and center of mass coordinates

$$y = \frac{1}{2}(\eta_1' + \eta_2'), \qquad z = \frac{1}{2}(\zeta_1' + \zeta_2'), \tag{111}$$

we obtain

$$\overline{S^2} = \overline{\mu^2} \int_0^{L'}\int_0^{L'}\int_{-\infty}^{+\infty}\int\int\int \Phi_1\left[L' - \xi_1', \quad \sqrt{(\frac{\eta}{2} + y)^2 + (\frac{\zeta}{2} + z)^2}\right] \tag{112}$$

$$\times\, \Phi_1\left[L' - \xi_1', \quad \sqrt{(\frac{\eta}{2} - y)^2 + (\frac{\zeta}{2} - z)^2}\right] N(r')d\xi_1'd\xi_2'd\eta d\zeta dydz.$$

$$\overline{B^2} = \overline{\mu^2} \int_0^{L'}\int_0^{L'}\int_{-\infty}^{+\infty}\int\int\int \Phi_2\left[L' - \xi_1', \quad \sqrt{(\frac{\eta}{2} + y)^2 + (\frac{\zeta}{2} + z)^2}\right] \tag{113}$$

$$\times\, \Phi_2\left[L' - \xi_2', \quad \sqrt{(\frac{\eta}{2} - y)^2 + (\frac{\zeta}{2} - z)^2}\right] N(r')d\xi_1'd\xi_2'd\eta d\zeta dydz.$$

We can carry out the integration with respect to the variables y and z in these formulas. From the evaluation of integrals of similar form given in Appendix II, it follows that

$$\int_{-\infty}^{+\infty}\int \Phi_1\left[L' - \xi_1', \quad \sqrt{(\frac{\eta}{2} + y)^2 + (\frac{\zeta}{2} + z)^2}\right] \tag{114}$$

$$\times\, \Phi_1\left[L' - \xi_2', \quad \sqrt{(\frac{\eta}{2} - y)^2 + (\frac{\zeta}{2} - z)^2}\right] dydz$$

$$= \frac{1}{2}\left(\Phi_1(\xi_1' - \xi_2', \rho) + \Phi_1\left[2L' - (\xi_1' + \xi_2'), \rho\right]\right).$$

$$\iint_{-\infty}^{+\infty} \Phi_2 \left[L' - \xi_1', \sqrt{(\frac{\eta}{2} + y)^2 + (\frac{\zeta}{2} + z)^2}\right] \times \tag{115}$$

$$\times \Phi_2 \left[L' - \xi_2', \sqrt{(\frac{\eta}{2} - y)^2 + (\frac{\zeta}{2} - z)^2}\right] dydz$$

$$= \frac{1}{2} \left(\Phi_1(\xi_1' - \xi_2', \rho) - \Phi_1\left[2L' - (\xi_1' + \xi_2'), \rho\right]\right),$$

where $\rho^2 = \sqrt{\eta^2 + \zeta^2}$.

After the integration, Eqs. (112) and (113) become

$$\overline{S^2} = \frac{1}{2} \overline{\mu^2} (I_1 + I_2), \tag{116}$$

$$\overline{B^2} = \frac{1}{2} \overline{\mu^2} (I_1 - I_2), \tag{117}$$

where

$$I_1 = \int_0^{L'}\int_0^{L'}\iint_{-\infty}^{+\infty} \Phi_1(\xi_1' - \xi_2', \rho)\, N(r')d\xi_1'd\xi_2'd\eta d\zeta, \tag{118}$$

$$I_2 = \int_0^{L'}\int_0^{L'}\iint_{-\infty}^{+\infty} \Phi_1 \left[2L' - (\xi_1' + \xi_2'), \rho\right] N(r')d\xi_1'd\xi_2'd\eta d\zeta. \tag{119}$$

Further simplification of the integrals I_1 and I_2 is possible in the case where the distance is large compared to the correlation distance ($L \gg a$). Introducing the relative coordinate $\xi = \xi_1' - \xi_2'$ and the center of mass coordinate $x = \frac{1}{2}(\xi_1' + \xi_2')$, we are justified in this case in integrating with respect to ξ from the limits $-\infty$ to $+\infty$. Then we obtain

$$I_1 = \int_0^{L'} dx \iiint_{-\infty}^{+\infty} \Phi_1(\xi, \rho)\, N(r')d\xi d\eta d\zeta, \tag{120}$$

$$I_2 = \int_0^{L'} dx \iiint_{-\infty}^{+\infty} \Phi_1(2L' - 2x, \rho)\, N(r')d\xi d\eta d\zeta. \tag{121}$$

Since the integrand in (120) does not depend on x, we immediately get

$$I_1 = L' \iiint_{-\infty}^{+\infty} \Phi_1(\xi, \rho)\, N(r')d\xi d\eta d\zeta \tag{122}$$

$$= L' \iiint_{-\infty}^{+\infty} \frac{1}{2\pi\xi} \sin \frac{\rho^2}{2\xi}\, N(r')d\xi d\eta d\zeta.$$

In (121) it is also possible to carry out the integration with respect to x. Since only one factor in the integrand, namely $\Phi_1[2L' - 2x, \rho]$, depends on x, the problem reduces to calculating the integral

$$\int_0^{L'} \Phi_1[2L' - 2x, \rho]\, dx = \int_0^{L'} \frac{1}{2\pi(2L' - 2x)} \sin \frac{\rho^2}{2(2L' - 2x)}\, dx.$$

By the substitution

$$z = \frac{\rho^2}{2(2L' - 2x)}$$

it is reduced to the form

$$\frac{1}{4\pi} \int_{\rho^2/4L'}^{\infty} \frac{\sin z}{z}\, dz = - \frac{1}{4\pi} \operatorname{si} \left(\frac{\rho^2}{4L'}\right)$$

(see the definition of the integral sine [39]). Thus

$$I_2 = -\iiint_{-\infty}^{+\infty} \frac{1}{4\pi}\, \mathrm{si}\left(\frac{\rho^2}{4L'}\right) N(r')\, d\xi\, d\eta\, d\zeta. \tag{123}$$

If in (122) and (123) we go over to polar coordinates (ρ,ϕ) in the (η,ζ) plane, and if we bear in mind that the correlation coefficient $N(r')$ is an even function of ξ, we obtain

$$I_1 = 2L' \int_0^\infty d\xi \int_0^\infty \frac{1}{\xi} \sin \frac{\rho^2}{2\xi}\, N(r')\rho\, d\rho, \tag{124}$$

$$I_2 = -\int_0^\infty d\xi \int_0^\infty \mathrm{si}\left(\frac{\rho^2}{4L'}\right) N(r')\rho\, d\rho. \tag{125}$$

Eq. (124) can be simplified further. Instead of ρ we introduce the new variable $q = \rho^2/2\xi$. Then the second integral in (124) can be written as follows:

$$\int_0^\infty \sin q \cdot N(r')\, dq. \tag{126}$$

We now integrate (126) by parts twice, i.e.

$$\int_0^\infty \sin q \cdot N(r')\, dq = \Big|_0^\infty - \cos q \cdot N(r') + \int_0^\infty \cos q\, \frac{\partial N(r')}{\partial q}\, dq \tag{127}$$

$$= N(\xi,0,0) + \Big|_0^\infty \sin q\, \frac{\partial N}{\partial q} - \int_0^\infty \sin q\, \frac{\partial^2 N(r')}{\partial q^2}\, dq$$

$$= N(\xi,0,0) - \int_0^\infty \sin q\, \frac{\partial^2 N(r')}{\partial q^2}\, dq.$$

As ρ changes, the correlation coefficient falls off from its maximum value of $N(\xi,0,0)$ at $\rho = 0$ to values which are effectively zero at distances ρ of the order of the correlation distance $a'(a' = ka)$. Correspondingly, the variable q changes to a quantity of the order of magnitude $a'^2/2\xi$, and since the region of appreciable values of ξ also does not exceed a', the change in q will not be less that $a'/2$, i.e.

$$q \gtrsim a'.$$

In view of this we have

$$\frac{\partial^2 N(r')}{\partial q^2} \sim \frac{1}{q^2} N(\xi,0,0) \lesssim \frac{N(\xi,0,0)}{a'^2} . \tag{129}$$

Therefore, the integral in the right hand side of (127) is of order $\frac{1}{a'}N(\xi,0,0)$. In the case under consideration, namely large-scale inhomogeneities $(a' \gg 1)$, we can neglect this integral as compared with the first term $N(\xi,0,0)$. Then we obtain

$$\int_0^\infty \sin q \cdot N(r')dq = N(\xi,0,0). \tag{130}$$

The integral I_1 (122) reduces to the form

$$I_1 = 2L' \int_0^\infty N(\xi,0,0)d\xi. \tag{131}$$

By introducing the variable

$$\nu = \frac{\rho^2}{4L'} \tag{132}$$

the integral I_2 (123) reduces to the form

$$I_2 = - 2L' \int_0^\infty d\xi \int_0^\infty \text{si } \nu \cdot N(r')d\nu. \tag{133}$$

Finally, for the mean square amplitude and phase fluctuations we get the following formulas:

$$\overline{S^2} = \overline{\mu^2}\, L' \int_0^\infty d\xi \left[N(\xi,0,0) - \int_0^\infty \mathrm{si}\ \nu\cdot N(r')d\nu\right], \tag{134}$$

$$\overline{B^2} = \overline{\mu^2}\, L' \int_0^\infty d\xi \left[N(\xi,0,0) + \int_0^\infty \mathrm{si}\ \nu\cdot N(r')d\nu\right]. \tag{135}$$

Eqs. (134) and (135) give the general solution of the problem of amplitude and phase fluctuations for the case of large-scale fluctuations ($ka \gg 1$) under the additional condition that the distance is large compared to the scale of the inhomogeneities ($L \gg a$). The form of the solution depends in an essential way on the size of the dimensionless parameter $D = 4L/ka^2$, defined as the ratio of the size of the first Fresnel zone to the scale of the inhomogeneities. Following Gorelik [51] we shall henceforth call this parameter the wave parameter.

Eqs. (134) and (135) can be studied without specifying the form of the correlation coefficient $N(r')$ in the following two cases which correspond to limiting values of the wave parameter: 1) $D \gg 1$, and 2) $D \ll 1$. This study reduces to an evaluation of the integral

$$I = \int_0^\infty \mathrm{si}\ \nu\cdot N(r')d\nu \tag{136}$$

in the two limiting cases mentioned.

20. <u>The Region of Large Values of the Wave Parameter (Fraunhofer Diffraction).</u> The integrand in (136) is different from zero for values of r' which do not exceed the correlation distance a' in order of magnitude. In this region of relevant values of r', the argument ν of the integral sine does not exceed the value $ka^2/4L$, i.e. does not exceed the value $1/D$

$$\nu \sim \frac{1}{D}. \tag{137}$$

In the case of a large wave parameter, ν will be small. Therefore

$$\mathrm{si}\ \nu \sim -\frac{\pi}{2}.$$

Then Eq. (136) takes the form

$$I = -\frac{\pi}{2} \int_0^\infty N(r')d\nu. \tag{138}$$

The integrand $N(r')$ has its maximum value $N(\xi,0,0)$ for $\nu = 0$ ($\eta = \zeta = 0$). From this maximum value $N(r')$ falls off practically to zero in a distance $\nu \sim 1/D$, so that

$$I \sim \frac{1}{D} N(\xi,0,0). \tag{139}$$

The integral I can be neglected compared to $N(\xi,0,0)$. Then Eqs. (134) and (135) can be written as one formula

$$\overline{S^2} = \overline{B^2} = \overline{\mu^2}L' \int_0^\infty N(\xi,0,0)d\xi \tag{140}$$

or, if we go over to the dimensional variables L and $\xi^* = \xi/k$

$$\overline{S^2} = \overline{B^2} = \overline{\mu^2}k^2L \int_0^\infty N(\xi^*,0,0)d\xi^*. \tag{140a}$$

Thus, in the case of a large wave parameter ($D \gg 1$) the mean square amplitude and phase fluctuations are the same and increase in proportion to the distance. For example, specifying the correlation coefficient in the form $N(\xi) = e^{-\xi^2/a^2}$, we obtain

$$\overline{S^2} = \overline{B^2} = \frac{\sqrt{\pi}}{2} \overline{\mu^2}k^2aL. \tag{141}$$

Mintzer [21], in studying pressure fluctuations of spherical wave sound pulses, used the method of small perturbations (in the usual form) and obtained the following formula:

$$\frac{\overline{|\Delta p|^2}}{|\bar{p}|^2} = 2\overline{\mu^2}k^2L \int_0^\infty N(\xi,0,0)d\xi .$$

The right hand side of this formula is twice as large as the right hand side of Eq. (140a). The reason for this difference is that the fluctuation of the field (pressure) is caused not only by amplitude fluctuations but also by phase fluctuations, i.e.

$$\frac{\overline{|\Delta p|^2}}{|\bar{p}|^2} = \overline{B^2} + \overline{S^2}.$$

In fact, the field can always be represented in the form

$$p = e^{B+iS+ikx}.$$

Expanding this expression in powers of the small quantities B and S and retaining only linear terms in the expansion, we find

$$p = (1 + B + iS)\, e^{ikx}.$$

Averaging the field over an ensemble of media, we obtain

$$\bar{p} = e^{ikx}$$

for the mean field. Moreover, we have

$$\Delta p = p - \bar{p} = (B + iS)\, e^{ikx},$$

from which it follows that

$$\frac{\overline{|\Delta p|^2}}{|\overline{p}|^2} = \overline{B^2} + \overline{S^2} .$$

Since $\overline{B^2} = \overline{S^2}$ in the region of large values of the wave parameter, we have

$$\frac{\overline{|\Delta p|^2}}{|\overline{p}|^2} = 2\overline{B^2} ,$$

i.e., the field fluctuations are twice as large as the amplitude (level) fluctuations.

In a later paper Mintzer [22] found a criterion for the applicability of the first approximation in the method of small perturbations, by requiring that the second approximation be small compared to the first approximation. This criterion is

$$\overline{\mu^2}k^2aL \ll 1,$$

and agrees with the criterion indicated in Section 13 for the applicability of the scattering formula. This restriction does not apply to Eqs. (140) and (141), which were obtained by a different method.

In studying the problem of the amplitude and phase fluctuations of a sound wave propagated in a turbulent atmosphere, Tatarski [32] used a scattering formula which is valid for large values of the wave parameter (Fraunhofer approximation). Therefore the formulas which he obtained for the mean square amplitude and phase fluctuations are suitable only in the case of a large wave parameter*.

<u>21. The Region of Small Values of the Wave Parameter.</u> As explained in the preceding section, the region of relevant values of the variable ν is of order $1/D$. In the case of a small wave parameter ($D \ll 1$) the region of relevant values of the variable ν extends to large values of ν ($\nu \gg 1$). The problem reduces to evaluating the asymptotic value of the integral I in

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* In a later paper Tatarski [56] studied the same problem in a more general way.

in the limiting case of large ν. To do this we integrate (136) by parts

$$I = \Big|_0^\infty (\nu \operatorname{si} \nu + \cos \nu)\, N(r') - \int_0^\infty (\nu \operatorname{si} \nu + \cos \nu)\, \frac{\partial N(r')}{\partial \nu}\, d\nu \tag{142}$$

$$= - N(\xi,0,0) - \int_0^\infty (\nu \operatorname{si} \nu + \cos \nu)\, \frac{\partial N(r')}{\partial \nu}\, d\nu .$$

To estimate the asymptotic value of the expression $\nu \operatorname{si} \nu + \cos \nu$ and certain other expressions, it is necessary to have an asymptotic expression for the integral sine. We can get such an expression from the asymptotic expression for the exponential integral Ei [39]

$$\operatorname{Ei}(-x) = e^{-x} \sum_{k=1}^{k=n} (-1)^k \frac{(k-1)!}{x^k} + R_n, \tag{143}$$

where

$$|R_n| < \frac{n!}{|x|^{n+1} \cos \phi/2}, \quad x = |x|\, e^{i\phi}, \ \phi^2 < \pi^2,$$

and from the formula

$$\operatorname{Ei}(-ix) = \operatorname{ci} x - i \operatorname{si} x. \tag{143a}$$

To evaluate $x \operatorname{si} x + \cos x$ it is sufficient to take two terms of the series (143). Then we get

$$\operatorname{si} x \sim - \frac{\cos x}{x} - \frac{\sin x}{x^2}, \tag{144}$$

$$x \operatorname{si} x + \cos x \sim - \frac{\sin x}{x}. \tag{145}$$

The derivative $\partial N(r')/\partial \nu$ is of order $DN(\xi,0,0)$ and the integrand in (142) is of the order $D^2 N(\xi,0,0)$. Therefore, the integral in (142) can be neglected compared to $N(\xi,0,0)$. Then

using (134) we obtain

$$\overline{S^2} = 2\overline{\mu^2}L' \int_0^\infty N(\xi,0,0)d\xi, \tag{146}$$

for the mean square phase fluctuation, or going over to the dimensional quantities L and ξ*

$$\overline{S^2} = 2\overline{\mu^2}k^2 L \int_0^\infty N(\xi^*,0,0)d\xi^*. \tag{146a}$$

Eq. (146) differs from the analogous formula (140) for the case of a large wave parameter by the factor 2. The mean square phase fluctuation grows in proportion to the distance both in the case of a large wave parameter and in the case of a small wave parameter.

Turning our attention now to the amplitude fluctuations, we note that in substituting (142) into (135) the leading terms $N(\xi,0,0)$ and $-N(\xi,0,0)$ cancel each other. This compels us to consider terms of a higher order of smallness with respect to the parameter D. Then we obtain from (135)

$$\overline{B^2} = -\overline{\mu^2}L' \int_0^\infty d\xi \int_0^\infty (\nu \text{ si } \nu + \cos \nu) \frac{\partial N(r')}{\partial \nu} d\nu. \tag{147}$$

Thus, in the case of a small wave parameter ($D \ll 1$) the amplitude fluctuations are an order of magnitude (with respect to the parameter D) smaller than the phase fluctuations. Since we wish to get a more suitable expression for the amplitude fluctuations, we integrate (147) by parts

$$\int_0^\infty (\nu \text{ si } \nu + \cos \nu) \frac{\partial N(r')}{\partial \nu} d\nu = \Big|_0^\infty \frac{1}{2} \left[\nu(\nu \text{ si } \nu + \cos \nu) + \sin \nu\right] \frac{\partial N(r')}{\partial \nu} \tag{148}$$

$$- \int_0^\infty \frac{1}{2} \left[\nu(\nu \text{ si } \nu + \cos \nu) + \sin \nu\right] \frac{\partial^2 N(r')}{\partial \nu^2} d\nu =$$

$$= - \int_0^{\infty} \left(\frac{1}{6}\, \nu\, [\nu(\nu \text{ si } \nu + \cos \nu) + \sin \nu] - \frac{1}{3} \cos \nu\right) \frac{\partial^2 N}{\partial \nu^2}$$

$$+ \int_0^{\infty} \left(\frac{1}{6}\, \nu[\nu(\nu \text{ si } \nu + \cos \nu) + \sin \nu] - \frac{1}{3} \cos \nu\right) \frac{\partial^3 N}{\partial \nu^3}\, d\nu$$

$$= - \frac{1}{3} \left[\frac{\partial^2 N(r')}{\partial \nu^2}\right]_{(\xi,0,0)} + \int_0^{\infty} \left\{\frac{1}{6}\, \nu[\nu(\nu \text{ si } \nu + \cos \nu) + \sin \nu] - \frac{1}{3} \cos \nu\right\} \frac{\partial^3 N}{\partial \nu^3}\, d\nu.$$

We can get an asymptotic expression for the curly brackets if we keep four terms in the expansion (143). Then we get

$$\text{si } \nu \sim - \left(\frac{1}{\nu} - \frac{2}{\nu^3}\right) \cos \nu - \left(\frac{1}{\nu^2} - \frac{6}{\nu^4}\right) \sin \nu, \tag{149}$$

$$\frac{1}{6}\, \nu[\nu(\nu \text{ si } \nu + \cos \nu) + \sin \nu] - \frac{1}{3} \cos \nu \sim \frac{\sin \nu}{\nu}\,. \tag{150}$$

The integral in (148) is of order $D^3N(\xi,0,0)$, and the term $\frac{1}{3}\left[\frac{\partial^2 N}{\partial \nu^2}\right]_{(\xi,0,0)}$ is of order $D^2N(\xi,0,0)$. Keeping the leading term, we obtain

$$\int_0^{\infty} (\nu \text{ si } \nu + \cos \nu) \frac{\partial N(r')}{\partial \nu}\, d\nu = -\frac{1}{3} \left[\frac{\partial^2 N(r')}{\partial \nu^2}\right]_{(\xi,0,0)}. \tag{151}$$

Bearing in mind that $\nu = (\eta^2 + \zeta^2)/4L'$, we can express $\left[\frac{\partial^2 N(r')}{\partial \nu^2}\right]_{(\xi,0,0)}$ in terms of derivatives with respect to the variables η and ζ. In fact

$$\frac{\partial^2 N}{\partial \eta^2} = \frac{1}{2L'}\frac{\partial N}{\partial \nu} + \frac{\eta^2}{4L'^2}\frac{\partial^2 N}{\partial \nu^2}, \quad \frac{\partial^2 N}{\partial \zeta^2} = \frac{1}{2L'}\frac{\partial N}{\partial \nu} + \frac{\zeta^2}{4L'^2}\frac{\partial^2 N}{\partial \nu^2}, \tag{152}$$

$$\nabla^2 N = \frac{1}{L'}\frac{\partial N}{\partial \nu} + \frac{\nu}{L'}\frac{\partial^2 N}{\partial \nu^2},$$

where

$$\nabla^2 = \frac{\partial^2}{\partial \eta^2} + \frac{\partial^2}{\partial \zeta^2}$$

is the transverse Laplacian. Moreover

$$\nabla^2\nabla^2 N = \left(\frac{1}{L'}\frac{\partial}{\partial \nu} + \frac{\nu}{L'}\frac{\partial^2}{\partial \nu^2}\right)\left(\frac{1}{L'}\frac{\partial N}{\partial \nu} + \frac{\nu}{L'}\frac{\partial^2 N}{\partial \nu^2}\right) \tag{153}$$

$$= \frac{2}{L'^2}\frac{\partial^2 N}{\partial \nu^2} + (\text{terms containing the factor } \nu).$$

Therefore

$$\frac{1}{3}\left[\frac{\partial^2 N(r')}{\partial \nu^2}\right]_{(\xi,0,0)} = \frac{1}{6} L'^2 \left[\nabla^2\nabla^2 N\right]_{\eta=\zeta=0}. \tag{154}$$

Finally, Eq. (147) takes the following form:

$$\overline{B^2} = \frac{1}{6}\overline{\mu^2} L'^3 \int_0^\infty \left[\nabla^2\nabla^2 N\right]_{\eta=\zeta=0} d\xi. \tag{155}$$

In going over to dimensional quantities it preserves the same form, i.e.

$$\overline{B^2} = \frac{1}{6}\overline{\mu^2} L^3 \int_0^\infty \left[\nabla^2\nabla^2 N\right]_{\eta^*=\zeta^*=0} d\xi^*. \tag{155a}$$

Eqs. (146a) and (155a) coincide with Eqs. (69) and (81) of Part I, which were obtained using the ray representation. Therefore, the case of a small wave parameter corresponds to the geometrical approximation. Ellison [31] showed this in his study of the formula for the mean square intensity fluctuations, and Tatarski [33] also showed this in his study of the formula for the (unaveraged) amplitude and phase fluctuations. In the geometrical approximation, the mean square amplitude fluctuation grows in proportion to the cube of the distance.

It is easy to give a qualitative explanation for the difference in the distance dependence of the mean square amplitude fluctuations in the regions of small and large values of the wave parameters. In the first case, the dimensions of the first Fresnel zone are small compared to the scale of the inhomogeneities. The deviations of the refractive index μ from its mean value within the zone all have the same sign. Therefore, all the waves scattered by the different elements of the first zone arrive at the observation point in phase, and the fluctuations increase rapidly with distance (like L^3). In the second case, the dimensions of the first Fresnel zone are large compared to the scale of the inhomogeneities. The deviations of the refractive index μ from its mean value have different signs at different points of the zone. Therefore, all of the elementary waves do not arrive in phase at the observation point. They partially interfere, and as a result the fluctuations grow more slowly with distance (like L).

22. The Region of Intermediate Values of the Wave Parameter. We can study the solutions (134) and (135) in the intermediate region by specifying the form of the correlation coefficient. Let us set

$$N = \exp\left[-\frac{\xi^2+\eta^2+\zeta^2}{a'^2}\right]$$

and introduce the variable ν defined by (132). Then we obtain

$$N = \exp\left[-\frac{\xi^2}{a'^2}-\frac{4L'\nu}{a'^2}\right] . \tag{158}$$

Substituting (158) in (134) and (135), we reduce the problem to evaluating the tabulated integrals [39]

$$\int_0^\infty N(\xi,0,0)d\xi = \int_0^\infty e^{-\xi^2/a'^2} d\xi = \frac{\sqrt{\pi}}{2} a',$$

$$\int_0^\infty \mathrm{si}\ \nu\ e^{-4L'\nu/a'^2} d\nu = -\frac{1}{4L'/a'^2} \text{ arc tan } \frac{4L'}{a'^2} = -\frac{1}{D} \text{ arc tan } D,$$

where

$$D = \frac{4L'}{a'^2} = \frac{4L}{ka^2}$$

is the wave parameter. Eqs. (134) and (135) become

$$\overline{S^2} = \frac{\sqrt{\pi}}{2}\ \overline{\mu^2}k^2 aL(1 + \frac{1}{D} \text{ arc tan } D), \tag{159}$$

$$\overline{B^2} = \frac{\sqrt{\pi}}{2}\ \overline{\mu^2}k^2 aL(1 - \frac{1}{D} \text{ arc tan } D). \tag{160}$$

Obukhov [12] obtained these formulas in this form. In the limiting cases of large and small wave parameters they go over into the corresponding formulas of Sections 21 and 22.

All that we have said about fluctuations applies equally to electromagnetic waves. In scattering by large-scale inhomogeneities the important role is played by the region of space located in the immediate neighborhood of the straight line connecting the transmitter and the receiver. In this region the angle X between the direction of the electric vector and the scattering direction is close to a right angle and the factor $\sin^2 X$ is close to unity, as a result of which the distinction between longitudinal and transverse waves disappears. The problem of fluctuations of the electromagnetic field caused by a turbulent atmosphere was investigated in the work of Rice [38], Megaw [18], Voge [20] and others.

Chapter VI

CORRELATION OF FLUCTUATIONS

The mean square amplitude and phase fluctuations still do not give a complete characterization of the statistical properties of the wave field. The statistical properties of the fluctuations of the wave field can be characterized more completely by using correlation functions. However, the question of the correlation of the fluctuations of the basic characteristics of the wave field, of both theoretical and practical interest, has not received much attention. In this connection, three questions arise naturally:

1. Is there correlation between the amplitude and phase fluctuations at the receiver?

2. How does the autocorrelation between amplitude (or phase) fluctuations depend on the distance between receivers?

3. What form has the time autocorrelation function for amplitude (or phase) fluctuations at the receiver?

As far as we know, the first question has not been considered before; we answer it here. The second question has been examined by some investigators [6,14], but only in the ray approximation for the case where both receivers are situated in a plane perpendicular to the direction of wave propagation (transverse autocorrelation). Moreover, they have not discovered the simple law to the effect that the transverse autocorrelation between amplitude (or phase) fluctuations extends over approximately the same distance as the correlation between the random inhomogeneities of the medium itself, if the scale of the latter is large compared to the wavelength. Here we show that this simple law is valid both in the region of small values of the wave parameter (the ray approximation) and in the region of large and intermediate values of the wave parameter. Moreover, we show that the longitudinal correlation extends over a distance which is considerably larger than the transverse correlation. In fact, if the distance between the receivers does not exceed the distance over which the ray approach is appropriate, then the amplitude (or phase) fluctuations are practically completely correlated.

The third question concerning the time autocorrelation was investigated by Mintzer [23],

who assumed that the correlation coefficient of the refractive index fluctuations can be factored into two terms, one depending only on the coordinate differences and the other depending only on the time difference, i.e. that the correlation function has the form

$$N(\xi_1 - \xi_2, \eta_1 - \eta_2, \zeta_1 - \zeta_2)\, M(t_1 - t_2).$$

This assumption is not justified in the case where the time change of the refractive index is caused by the motion of the inhomogeneities, which occurs in the presence of drift and convection. In this book we consider the question of how motion of the inhomogeneities affects the time correlation properties of the acoustic field at the receiving point.

23. Correlation of the Amplitude and Phase Fluctuations at the Receiver. We begin by studying the cross correlation of the amplitude and phase fluctuations at the receiver. To do this we determine the form of the correlation function $\overline{BS}$, using Eqs. (106) and (107). Multiplying these equations and averaging them, we obtain

$$\overline{BS} = \overline{\mu^2} \int_0^{L'}\int_0^{L'}\int\int\int_{-\infty}^{+\infty}\int \Phi_1(L' - \xi_1', \rho_1')\, \Phi_2(L' - \xi_2', \rho_2')\, N(r')\, d\xi_1' d\xi_2' d\eta_1' d\eta_2' d\zeta_1' d\zeta_2'. \tag{161}$$

Considering only the case of a statistically isotropic medium, we shall suppose that the correlation coefficient N depends only on the magnitude r. Introducing relative and center of mass coordinates in (110) and (111), we obtain

$$\overline{BS} = \overline{\mu^2} \int_0^{L'}\int_0^{L'}\int\int\int_{-\infty}^{+\infty}\int \Phi_1\left[L' - \xi_1', \quad \sqrt{\left(\frac{\eta}{2} + y\right)^2 + \left(\frac{\zeta}{2} + z\right)^2}\right] \times \tag{162}$$

$$\times\ \Phi_2\left[L' - \xi_2', \ \sqrt{\left(\frac{\eta}{2} - y\right)^2 + \left(\frac{\zeta}{2} - z\right)^2}\right] N(r')\, d\xi_1' d\xi_2' d\eta\, d\zeta\, dy\, dz.$$

In this formula we can carry out the integration with respect to the variables y and z (see Appendix II):

$$\iint\limits_{-\infty}^{+\infty} \Phi_1\left[L' - \xi_1', \sqrt{\left(\frac{\eta}{2} + y\right)^2 + \left(\frac{\zeta}{2} + z\right)^2}\right] \times \tag{163}$$

$$\times\, \Phi_2\left[L' - \xi_2', \sqrt{\left(\frac{\eta}{2} - y\right)^2 + \left(\frac{\zeta}{2} - z\right)^2}\right] dydz$$

$$= \frac{1}{2}\left(\Phi_2\left[2L' - (\xi_1' + \xi_2'), \rho\right] + \Phi_2(\xi_1' - \xi_2', \rho)\right) ,$$

where $\rho = \sqrt{\eta^2 + \zeta^2}$. Then we obtain

$$\overline{BS} = \frac{1}{2}\,\overline{\mu^2} \int\limits_0^{L'}\int\limits_0^{L'}\iint\limits_{-\infty}^{+\infty} \Phi_2\left[2L' - (\xi_1' + \xi_2'), \rho\right] N(r')d\xi_1'd\xi_2'd\eta d\zeta \tag{164}$$

$$- \frac{1}{2}\,\overline{\mu^2} \int\limits_0^{L'}\int\limits_0^{L'}\iint\limits_{-\infty}^{+\infty} \Phi_2(\xi_1' - \xi_2', \rho)\, N(r')d\xi_1'd\xi_2'd\eta d\zeta .$$

Further simplification of Eq. (164) is possible in the case L >> a. Introducing the coordinates $2x = \xi_1' + \xi_2'$ and $\xi = \xi_1' - \xi_2'$, in this case we can integrate with respect to ξ between the limits $-\infty$ and $+\infty$. Eq. (164) can be written in the form

$$\overline{BS} = \frac{1}{2}\,\overline{\mu^2} \int\limits_0^{L'} dx \iiint\limits_{-\infty}^{+\infty} \Phi_2\left[2L' - 2x, \rho\right] N(r')d\xi d\eta d\zeta \tag{165}$$

$$- \frac{1}{2}\,\overline{\mu^2} \int\limits_0^{L'} dx \iiint\limits_{-\infty}^{+\infty} \Phi_2(\xi, \rho)\, N(r')d\xi d\eta d\zeta .$$

Since $N(r')$ is an even function of ξ, while $\Phi_2(\xi,\rho)$ is an odd function of ξ, the second integral vanishes. In the first integral we can integrate with respect to x. Thus the problem reduced to calculating the integral

$$\int_0^{L'} \Phi_2[2L' - 2x, \rho]\,dx = \int_0^{L'} \frac{1}{2\pi(2L' - 2x)} \cos \frac{\rho^2}{2(2L' - 2x)}\,dx.$$

The substitution $z = \frac{\rho^2}{2(2L' - 2x)}$ reduces it to the form

$$\frac{1}{4\pi} \int_{\rho^2/4L'}^{\infty} \frac{\cos z}{z}\,dz = -\frac{1}{4\pi}\,\mathrm{ci}\left(\frac{\rho^2}{4L'}\right)$$

(see the definition of the integral cosine [39]). Eq. (165) then takes the form

$$\overline{BS} = -\frac{1}{8\pi}\,\overline{\mu^2} \iiint_{-\infty}^{+\infty} \mathrm{ci}\left(\frac{\rho^2}{4L'}\right) N(r')\,d\xi\,d\eta\,d\zeta\,. \tag{166}$$

Transforming to polar coordinates (ρ,ϕ) in the (η,ζ) plane, we obtain

$$\overline{BS} = -\frac{1}{2}\,\overline{\mu^2} \int_0^{\infty} d\xi \int_0^{\infty} \mathrm{ci}\left(\frac{\rho^2}{4L'}\right) N(r')\,\rho\,d\rho. \tag{167}$$

Finally, we introduce as before the variable ν:

$$\nu = \frac{\rho^2}{4L'}\,. \tag{168}$$

Then

$$\overline{BS} = -\overline{\mu^2}\,L' \int_0^{\infty} d\xi \int_0^{\infty} \mathrm{ci}\ \nu\ \ N(r')\,d\nu. \tag{169}$$

Eq. (169) gives the general solution of the problem of the correlation of amplitude and phase fluctuations at the receiver.

Suppose the correlation coefficient has the form

$$N(r') = \exp\left[-\frac{\xi^2 + \eta^2 + \zeta^2}{a'^2}\right]. \tag{170}$$

According to (158) the problem reduces to calculating the integral

$$\int_0^\infty \mathrm{ci}\ \nu \exp\left[\frac{4L'}{a'^2}\nu\right] d\nu.$$

We find in the tables [39] that

$$\int_0^\infty \mathrm{ci}\ \nu \exp\left[-\frac{4L'}{a'^2}\nu\right] d\nu = -\frac{a'^2}{4L'} \log \sqrt{1 + \left(\frac{4L'}{a'^2}\right)^2} .$$

Finally we obtain

$$\overline{BS} = \frac{\sqrt{\pi}}{16}\ \overline{\mu^2} k^3 a^3 \log(1 + D^2). \tag{171}$$

In probability theory [44] the correlation coefficient is defined as the ratio of the correlation function of the fluctuations to their r.m.s. values. Applied to our case this means that the correlation coefficient R_{bs} of the amplitude and phase correlations has the form

$$R_{bs} = \frac{\overline{BS}}{\sqrt{\overline{B^2}}\sqrt{\overline{S^2}}} . \tag{172}$$

Using the formula (171) for the correlation function and formulas (159) and (160) for the mean square values of the amplitude and phase fluctuations, we finally get the following expression for the correlation coefficient R_{bs}:

$$R_{bs} = \frac{1}{2} \frac{\log (1 + D^2)}{\sqrt{D^2 - (\arc \tan D)^2}} . \tag{173}$$

At small distances ($D \ll 1$), when the ray approach is appropriate, Eq. (173) gives

$$R_{bs} \sim \frac{1}{2} \sqrt{\frac{3}{2}} \sim 0.6.$$

At large distances ($D \gg 1$), Eq. (173) takes the form

$$R_{bs} = \frac{\log D}{D} ,$$

i.e., the correlation coefficient falls off with distance and approaches zero. Thus, the correlation between amplitude and phase fluctuations which exists at small distances vanishes at large distances.

24. Longitudinal Autocorrelation of the Amplitude (or Phase) Fluctuations. We now consider the question of the autocorrelation of the amplitude (or phase) fluctuations at different receiving points. Let us assume that the receivers are situated along the direction of propagation. Let the coordinates of the receivers be $(L_1',0,0)$ and $(L_2',0,0)$. The amplitude and phase fluctuations are given by the formulas (106) and (107). In these formulas we have to put $L' = L_1'$ for the first receiver and $L' = L_2'$ for the second receiver. Therefore, the autocorrelation functions for the amplitude and phase fluctuations are given by the following formulas:

$$\overline{S_1 S_2} = \overline{\mu^2} \int_0^{L_1'}\int_0^{L_2'}\int\int\int\int_{-\infty}^{+\infty} \Phi_1(L_1' - \xi_1', \rho_1') \, \Phi_1(L_2' - \xi_2', \rho_2') \times \tag{174}$$

$$\times N(r') d\xi_1' d\xi_2' d\eta_1' d\eta_2' d\zeta_1' d\zeta_2' ,$$

$$\overline{B_1B_2} = \overline{\mu^2} \int_0^{L_1'}\int_0^{L_2'}\int_{-\infty}^{+\infty}\int\int\int\int \Phi_2(L_1' - \xi_1', \rho_1')\, \Phi_2(L_2' - \xi_2', \rho_2') \times \tag{175}$$

$$\times\, N(r')d\xi_1'd\xi_2'd\eta_1'd\eta_2'd\zeta_1'd\zeta_2' .$$

In these formulas the upper limit of the integration with respect to ξ_2' can also be set equal to L_1'. In fact, because of the directional character of the scattering, the waves scattered by the layer bounded by the planes $x = L_1$ and $x = L_2$ are incident on the second receiver but not on the first. Therefore, these waves can be neglected in the calculation of autocorrelation functions.

Setting the upper limit of the integration with respect to ξ_2' equal to L_1', we obtain

$$\overline{S_1S_2} = \overline{\mu^2} \int_0^{L_1'}\int_0^{L_1'}\int_{-\infty}^{+\infty}\int\int\int\int \Phi_1(L_1' - \xi_1', \rho_1')\, \Phi_1(L_2' - \xi_2', \rho_2') \times \tag{176}$$

$$\times\, N(r')d\xi_1'd\xi_2'd\eta_1'd\eta_2'd\zeta_1'd\zeta_2' ,$$

$$\overline{B_1B_2} = \overline{\mu^2} \int_0^{L_1'}\int_0^{L_1'}\int_{-\infty}^{+\infty}\int\int\int\int \Phi_2(L_1' - \xi_1', \rho_1')\, \Phi_2(L_2' - \xi_2', \rho_2') \times \tag{177}$$

$$\times\, N(r')d\xi_1'd\xi_2'd\eta_1'd\eta_2'd\zeta_1'd\zeta_2' .$$

Using Eqs. (110) and (111) to transform coordinates and using Eqs. (114) and (115) to carry out the integration, we find

$$\overline{S_1S_2} = \frac{1}{2}\, \overline{\mu^2}(I_1 + I_2), \tag{176a}$$

$$\overline{B_1B_2} = \frac{1}{2}\, \overline{\mu^2}(I_1 - I_2), \tag{177a}$$

where

$$I_1 = \int_0^{L_1'}\int_0^{L_1'}\int_{-\infty}^{+\infty}\int \Phi_1\left[L_2' - L_1' - (\xi_2' - \xi_1'),\ \rho\right] N(r')d\xi_1'd\xi_2'd\eta d\zeta, \tag{178}$$

$$I_2 = \int_0^{L_1'}\int_0^{L_1'}\int_{-\infty}^{+\infty}\int \Phi_1\left[L_2' + L_1' - (\xi_1' + \xi_2'),\ \rho\right] N(r')d\xi_1'd\xi_2'd\eta d\zeta. \tag{179}$$

These last formulas constitute a generalization of Eqs. (118) and (119) for the case $L_2' \neq L_1'$. The way in which they are manipulated further is analogous to the manipulation of Eqs. (118) and (119). Introducing the coordinates $\xi = \xi_2' - \xi_1'$ and $x = \frac{1}{2}(\xi_2' + \xi_1')$ and assuming $L \gg a$, we obtain

$$I_1 = L_1' \int_{-\infty}^{+\infty} d\xi \int_0^{\infty} \frac{1}{\Delta L' - \xi} \sin \frac{\rho^2}{2(\Delta L' - \xi)} N(r')\rho d\rho. \tag{180}$$

Eq. (180) is analogous to (124). By introducing the new variable

$$z = \frac{\rho^2}{2(L_2' + L_1' - 2x)}$$

the integration with respect to x in (179) is carried out as follows:

$$\int_0^{L_1'} \frac{1}{2\pi(L_2' + L_1' - 2x)} \sin \frac{\rho^2}{2(L_2' + L_1' - 2x)} dx$$

$$= \frac{1}{4\pi} \int_{\rho^2/2(L_2'+L_1')}^{\rho^2/2\Delta L'} \frac{\sin z}{z} dz = - \frac{1}{4\pi} \left(\mathrm{si} \frac{\rho^2}{2(L_2' + L_1')} - \mathrm{si} \frac{\rho^2}{2\Delta L'}\right) .$$

Consequently,

$$I_2 = - \int_0^{\infty} d\xi \int_0^{\infty} \left(si \; \frac{\rho^2}{2(L_2' + L_1')} - si \; \frac{\rho^2}{2\Delta L'} \right) N(r') \rho d\rho . \tag{181}$$

This formula is analogous to (125).

Suppose that the correlation coefficient $N(r')$ has the form (170). Introducing the variable $q = \rho^2/2(\Delta L' - \xi)$, we obtain

$$\int_0^{\infty} \frac{1}{\Delta L' - \xi} \sin \frac{\rho^2}{2(\Delta L' - \xi)} \exp\left[- \frac{\xi^2 + \eta^2 + \zeta^2}{a'^2} \right] \rho d\rho \tag{182}$$

$$= \exp\left[- \frac{\xi^2}{a'^2} \right] \int_0^{\infty} \sin q \exp\left[- \frac{2(\Delta L' - \xi)}{a'^2} x \right] dq = \frac{\exp\left[- \xi^2/a'^2\right]}{1 + \frac{4(\Delta L' - \xi)^2}{a'^4}} , \text{ for } \Delta L' > \xi .$$

For $\Delta L' < \xi$ we get the same result by setting $q = -\rho^2/2(\Delta L' - \xi)$. By (180) we have

$$I_1 = L_1' \int_{-\infty}^{+\infty} \frac{\exp\left[- \xi^2/a'^2\right] d\xi}{1 + \frac{4(\Delta L'^2 - 2\Delta L'\xi + \xi^2)}{a'^4}} . \tag{183}$$

Calculation of an integral like I_2 was encountered at the end of the preceding chapter (see p. 83). Repeating similar calculations, we obtain

$$I_2 = \sqrt{\pi} \, a' \, \frac{L_1' + L_2'}{2} \left[\frac{1}{\frac{2(L_1' + L_2')}{a'^2}} \arc \tan \frac{2(L_1' + L_2')}{a'^2} \right. \tag{184}$$

$$\left. - \frac{1}{\frac{2(L_1' + L_2')}{a'^2}} \arc \tan \frac{2\Delta L'}{a'^2} \right] .$$

receivers. Then

$$\frac{1}{2}(L_1' + L_2') \sim L_1',$$

and Eq. (184) takes the form

$$I_2 = \sqrt{\pi}\, a'L_1'\left[\frac{1}{D}\text{ arc tan } D - \frac{1}{D}\text{ arc tan }\left(\frac{1}{2}\frac{\Delta L'}{L_1'}D\right)\right], \tag{185}$$

where

$$D = \frac{4L_1'}{a'^2} = \frac{4L_1}{ka^2}.$$

In the region of large values of the wave parameter D the integral I_2 can be neglected compared to the integral I_1. Then we obtain

$$\overline{B_1B_2} = \overline{S_1S_2} = \frac{1}{2}\overline{\mu^2}\, L_1' \int_{-\infty}^{+\infty} \frac{\exp\left[-\xi^2/a'^2\right]d\xi}{1 + 4\dfrac{\Delta L'^2 - 2\Delta L'\xi + \xi^2}{a'^4}}. \tag{186}$$

If the distance $\Delta L'$ between the receivers is of the order of the correlation distance a', then the second term in the denominator can be neglected as compared with unity. In fact, the integrand differs from zero in a region of values of ξ of order a'. In this region of significant values of ξ the second term in the denominator will be of order

$$\frac{a'^2}{a'^4} = \frac{1}{a'^2} = \frac{1}{(ka)^2}$$

for ka $\gg$ 1. In this case ($\Delta L \sim a$), we obtain

$$\overline{B_1B_2} = \overline{S_1S_2} = \frac{\sqrt{\pi}}{2}\overline{\mu^2}\, k^2 aL,$$

which coincides with the formula (141) for the mean square amplitude and phase fluctuation. Thus, at a distance ΔL of the order of the correlation distance a of the refractive index,

both the amplitude and the phase fluctuations are practically completely correlated.

We consider next, therefore, distances ΔL which are large compared to the correlation distance a, i.e. $\Delta L \gg a'$. In the region of significant values of ξ the terms $\Delta L'\xi$ and ξ^2 have the orders of magnitude $\Delta L'a'$ and a'^2, respectively. Therefore, we can neglect these terms as compared with the term $(\Delta L')^2$. Then Eq. (186) becomes

$$\overline{B_1 B_2} = \overline{S_1 S_2} = \frac{\frac{\sqrt{\pi}}{2} \overline{\mu^2} k^2 aL}{1 + \left(\frac{2\Delta L}{ka^2}\right)^2} . \tag{187}$$

By (187) and (141) we obtain

$$R_b = R_s = \frac{1}{1 + \left(\frac{2\Delta L}{ka^2}\right)^2} \tag{188}$$

for the correlation coefficients

$$R_b = \frac{\overline{B_1 B_2}}{\overline{B^2}} , \quad R_s = \frac{\overline{S_1 S_2}}{\overline{S^2}} .$$

As ΔL grows, the correlation coefficients fall off. Taking for a correlation distance the distance ΔL_k at which the correlation coefficient is reduced by a factor of two, we obtain the formula

$$\frac{2\Delta L_k}{ka^2} = 1 \tag{189}$$

for the definition of the correlation distance. Comparing the left hand side of (189) with the expression for the wave parameter $D = 4L/ka^2$ we arrive at the following conclusion:

The longitudinal autocorrelation of amplitude (or phase) extends over the distance within which the ray approach is appropriate. This distance is many times larger than the correlation distance of the medium.

Because of what we have just shown, it is no longer necessary to consider the other limiting case, corresponding to small values of the wave parameter: in this region there is complete longitudinal autocorrelation of the amplitude and phase fluctuations. Moreover, it follows from the preceding section that in this region the amplitude and phase fluctuations are

completely cross-related.

25. Transverse Autocorrelation of the Amplitude and Phase Fluctuations. Suppose that both receivers lie in the plane $x = L$ at a distance ℓ from one another. Let the coordinates of the receivers be $(L,0,0)$ and $(L,0,\ell)$, respectively. The amplitude and phase fluctuations are given by Eqs. (106) and (107) both for the first receiver and for the second receiver. The difference is only that we must take $\rho_1' = \sqrt{\eta_1^2 + \zeta_1^2}$ instead of ρ' for the first receiver and $\rho_2' = \sqrt{\eta_2^2 + (\ell' - \zeta_2)^2}$ for the second receiver. Therefore, the autocorrelation functions for the amplitude and phase fluctuations are given by the following formulas:

$$\overline{S_1 S_2} = \overline{\mu^2} \int_0^{L'}\int_0^{L'}\int\int\int\int_{-\infty}^{+\infty} \Phi_1(L' - \xi_1', \rho_1') \Phi_1(L' - \xi_2', \rho_2') \times \tag{190}$$

$$\times N(r') d\xi_1' d\xi_2' d\eta_1' d\eta_2' d\zeta_1' d\zeta_2',$$

$$\overline{B_1 B_2} = \overline{\mu^2} \int_0^{L'}\int_0^{L'}\int\int\int\int_{-\infty}^{+\infty} \Phi_2(L' - \xi_1', \rho_1') \Phi_2(L' - \xi_2', \rho_2') \times \tag{191}$$

$$\times N(r') d\xi_1' d\xi_2' d\eta_1' d\eta_2' d\zeta_1' d\zeta_2'.$$

Using Eqs. (110) and (111) to go over to relative and center of mass coordinates, we obtain

$$\overline{S_1 S_2} = \overline{\mu^2} \int_0^{L'}\int_0^{L'}\int\int\int\int_{-\infty}^{+\infty} \Phi_1\left[L' - \xi_1',\ \sqrt{(\tfrac{\eta}{2} + y)^2 + (\tfrac{\zeta}{2} + z)^2}\right] \times \tag{192}$$

$$\times \Phi_1\left[L' - \xi_2',\ \sqrt{(\tfrac{\eta}{2} - y)^2 + (\ell' + \tfrac{\zeta}{2} - z)^2}\right] \times$$

$$\times N(r') d\xi_1' d\xi_2' d\eta d\zeta dy dz,$$

$$\overline{B_1B_2} = \overline{\mu^2} \int_0^{L'}\int_0^{L'}\int\int\int\int_{-\infty}^{+\infty} \Phi_2\left[L' - \xi_1',\ \sqrt{(\tfrac{\eta}{2} + y)^2 + (\tfrac{\zeta}{2} + z)^2}\right] \times \tag{193}$$

$$\times\, \Phi_2\left[L' - \xi_2',\ \sqrt{(\tfrac{\eta}{2} - y)^2 + (\ell' + \tfrac{\zeta}{2} - z)^2}\right] \times$$

$$\times\ N(r')d\xi_1'd\xi_2'd\eta d\zeta dydz.$$

The integration with respect to the variables y and z can be carried out in these formulas (see Appendix II):

$$\int\int_{-\infty}^{+\infty} \Phi_1\left[L' - \xi_1',\ \sqrt{(\tfrac{\eta}{2} + y)^2 + (\tfrac{\zeta}{2} + z)^2}\right] \times \tag{194}$$

$$\times\, \Phi_1\left[L' - \xi_2',\ \sqrt{(\tfrac{\eta}{2} - y)^2 + (\ell' + \tfrac{\zeta}{2} - z)^2}\right]dydz$$

$$= \frac{1}{2}\left(\Phi_1(\xi_1' - \xi_2',\ \rho) + \Phi_1\left[2L' - (\xi_1' + \xi_2'),\ \rho\right]\right)\ ,$$

$$\int\int_{-\infty}^{+\infty} \Phi_2\left[L' - \xi_1',\ \sqrt{(\tfrac{\eta}{2} + y)^2 + (\tfrac{\zeta}{2} + z)^2}\right] \times \tag{195}$$

$$\times\, \Phi_2\left[L' - \xi_2',\ \sqrt{(\tfrac{\eta}{2} - y)^2 + (\ell' + \tfrac{\zeta}{2} - z)^2}\right]dydz$$

$$= \frac{1}{2}\left(\Phi_1(\xi_1' - \xi_2',\ \rho) - \Phi_1\left[2L' - (\xi_1' + \xi_2')'\ \rho\right]\right)\ ,$$

where $\rho^2 = \eta^2 + (\zeta + \ell')^2$. Then we obtain

$$\overline{S_1S_2} = \frac{1}{2}\,\overline{\mu^2}(I_1 + I_2), \tag{196}$$

$$\overline{B_1 B_2} = \frac{1}{2}\, \overline{\mu^2}(I_1 - I_2), \tag{197}$$

where

$$I_1 = \int_0^{L'}\int_0^{L'}\int\int_{-\infty}^{+\infty} \Phi_1(\xi_1' - \xi_2', \rho)\; N(r')\, d\eta\, d\zeta\, d\xi_1'\, d\xi_2', \tag{198}$$

$$I_2 = \int_0^{L'}\int_0^{L'}\int\int_{-\infty}^{+\infty} \Phi_1\left[2L' - (\xi_1' + \xi_2'), \rho\right]\; N(r')\, d\eta\, d\zeta\, d\xi_1'\, d\xi_2'. \tag{199}$$

Eqs. (196) - (199) give in general form the solution to the problem of the amplitude (and phase) autocorrelation at different points of the plane $x = L$. They are identical in their structure to Eqs. (116) - (119) of the preceding chapter, which give the mean square amplitude and phase fluctuations at the receiver. The only difference is in the definition of ρ. Previously, ρ was defined by the formula $\rho^2 = \eta^2 + \zeta^2$, but now ρ is defined by the formula $\rho^2 = \eta^2 + (\zeta + \ell')^2$. This seemingly small difference leads to a destruction of axial symmetry, which makes it inappropriate to introduce polar coordinates in the (η, ζ) plane and makes it impossible to make a general study of the integrals (198) and (199) in the way that we studied the integrals (118) and (119) for the limiting cases of large and small values of the wave parameter. Instead we have to start with some explicit correlation coefficient $N(r')$ and carry out all the calculations in a somewhat different order than before. It can be assumed that the qualitative conclusions which are obtained for a special form of the correlation coefficient remain valid for another choice of the correlation coefficient.

If the distance is large compared to the correlation distance ($L \gg a$), then introducing the relative coordinate $\xi = \xi_1' - \xi_2'$ and the center of mass coordinate $x = \frac{1}{2}(\xi_1' + \xi_2')$, we can integrate with respect to ξ between the limits $-\infty$ and $+\infty$. Then we obtain

$$I_1 = \int_0^{L'} dx \int_{-\infty}^{+\infty} d\xi \int\int_{-\infty}^{+\infty} \Phi_1(\xi, \rho)\; N(r')\, d\eta\, d\zeta, \tag{200}$$

$$I_2 = \int_0^{L'} dx \int_{-\infty}^{+\infty} d\xi \iint_{-\infty}^{+\infty} \Phi_1(2L' - 2x, \rho)\, N(r') d\eta d\zeta . \tag{201}$$

Suppose the correlation coefficient has the form

$$N(r') = \exp\left[-\frac{\xi^2 + \eta^2 + \zeta^2}{a'^2}\right]. \tag{202}$$

By (101) the integrals I_1 and I_2 can be written in the following way:

$$I_1 = \int_0^{L'} dx \int_{-\infty}^{+\infty} d\xi \iint_{-\infty}^{+\infty} \frac{1}{2\pi\xi} \sin \frac{\eta^2 + (\zeta + \ell')^2}{2\xi} \exp\left[-\frac{\xi^2 + \eta^2 + \zeta^2}{a'^2}\right] d\eta d\zeta, \tag{203}$$

$$I_2 = \int_0^{L'} dx \int_{-\infty}^{+\infty} d\xi \iint_{-\infty}^{+\infty} \frac{1}{4\pi(L' - x)} \sin \frac{\eta^2 + (\zeta + \ell')^2}{4(L' - x)} \times \tag{204}$$

$$\times \exp\left[-\frac{\xi^2 + \eta^2 + \zeta^2}{a'^2}\right] d\eta d\zeta .$$

Consider now the calculation of the integral I_1. First of all we carry out the integration with respect to the variables η and ζ:

$$\iint_{-\infty}^{+\infty} \sin \frac{\eta^2 + (\zeta + \ell')^2}{2\xi} \exp\left[-\frac{\eta^2 + \zeta^2}{a'^2}\right] d\eta d\zeta \tag{205}$$

$$= \frac{1}{2i} \iint_{-\infty}^{+\infty} \exp\left[-\frac{\eta^2 + \zeta^2}{a'^2} + i\, \frac{\eta^2 + (\zeta + \ell')^2}{2\xi}\right] d\eta d\zeta$$

$$- \frac{1}{2i} \iint_{-\infty}^{+\infty} \exp\left[-\frac{\eta^2 + \zeta^2}{a'^2} - i\, \frac{\eta^2 + (\zeta + \ell')^2}{2\xi}\right] d\eta d\zeta$$

$$= \frac{1}{2i} \int_{-\infty}^{+\infty} \exp\left[-\left(\frac{1}{a'^2} - \frac{i}{2\xi}\right)\eta^2\right] d\eta \int_{-\infty}^{+\infty} \exp\left[-\frac{\zeta^2}{a'^2} + i\frac{(\zeta+\ell')^2}{2\xi}\right] d\zeta$$

+ the complex conjugate integral.

Moreover,

$$\int_{-\infty}^{+\infty} \exp\left[-\left(\frac{1}{a'^2} - \frac{i}{2\xi}\right)\eta^2\right] d\eta = \sqrt{\frac{\pi}{\frac{1}{a'^2} - \frac{i}{2\xi}}} .$$

In order to integrate with respect to ζ, we introduce a new variable ζ' by the substitution

$$\zeta' = \zeta - \frac{i\ell'}{\frac{2\xi}{a'^2} - i} \quad \text{or} \quad \zeta' = \zeta + c - id,$$

where

$$c = \frac{\ell'}{1 + \frac{4\xi^2}{a'^4}} , \quad d = \frac{\frac{2\ell'\xi}{a'^2}}{1 + \frac{4\xi^2}{a'^4}} . \tag{206}$$

The quantities c and d are finite for any value of ξ. The new variable ζ' varies from $-\infty - id$ to $+\infty - id$. The integral with respect to ζ can be transformed into the form

$$\exp\left[i\frac{\ell'^2/a'^2}{(2\xi/a'^2)-i}\right] \int_{-\infty-id}^{+\infty-id} \exp\left[-\left(\frac{1}{a'^2} - \frac{i}{2\xi}\right)\zeta'^2\right] d\zeta' . \tag{207}$$

The integration along the line parallel to the imaginary axis (a distance -id away from it) can be replaced by an integration along the real axis. Indeed, choose the contour of integration as shown in Fig. 4. Then by Cauchy's theorem

$$\int_{I} + \int_{II} + \int_{III} + \int_{IV} = 0 . \tag{208}$$

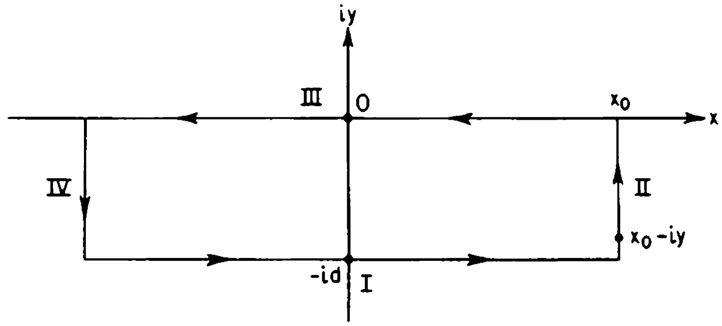

Fig. 4 The contour of integration for Eq. (208).

On the segments II and IV, where $\zeta' = \pm x_o - iy$, we have

$$\mathrm{Re}\left[\left(\frac{1}{a'^2} - \frac{i}{2\xi}\right) \zeta'^2\right] = \frac{x_o^2 - y^2}{a'^2} \mp \frac{x_o y}{\xi} .$$

Since $|y| \le |d|$, then $x_o y/\xi \le x_o d/\xi$, or by (206)

$$\frac{x_o y}{2\xi} \le \frac{\dfrac{\ell' x_o}{a'^2}}{1 + \dfrac{4\xi^2}{a'^4}} \le \frac{\ell' x_o}{a'^2} .$$

For sufficiently large x_o the term $x_o y/\xi$ can be neglected as compared with the term x_o^2/a'^2, i.e.

$$\mathrm{Re}\left[\left(\frac{1}{a'^2} - \frac{i}{2\xi}\right) \zeta'^2\right] \sim \frac{x_o^2}{a'^2} .$$

Therefore $\int_{II,IV} \to 0$ as $x_o \to \infty$, and from (208) we obtain in the limit

$$\int_{I} = - \int_{III} .$$

Integrating along the real axis in (207) we obtain[†]

$$\exp\left[\frac{\ell'^2/a'^2}{(2\xi/a'^2) - i}\right] \sqrt{\frac{\pi}{\frac{1}{a'^2} - \frac{i}{2\xi}}} \; .$$

By using (203) the integral I_1 can be written in the following way

$$I_1 = \frac{1}{2i} \int_0^{L'} dx \int_{-\infty}^{+\infty} d\xi \; e^{-\xi^2/a'^2} \left(\frac{e^{\gamma_1}}{(2\xi/a'^2)-i} - \frac{e^{\gamma_1^*}}{(2\xi/a'^2)+i} \right) , \tag{203a}$$

where

$$\gamma_1 = \frac{i\ell'^2/a'^2}{(2\xi/a'^2) - i}$$

(γ_1^* is the complex conjugate of γ_1). In just the same way we carry out the integration with respect to η and ζ in (204), obtaining

$$I_2 = \frac{1}{2i} \int_0^{L'} dx \int_{-\infty}^{+\infty} d\xi \; e^{-\xi^2/a'^2} \left(\frac{e^{\gamma_2}}{[4(L'-x)/a'^2] - i} - \frac{e^{\gamma_2^*}}{[4(L'-x)/a'^2] + i} \right) , \tag{209}$$

† The result can be obtained more quickly if in doing the integration in (205) we use the formula

$$\int_{-\infty}^{+\infty} \exp(-p\zeta^2 \pm q\zeta) d\zeta = \sqrt{\frac{\pi}{p}} \exp\left(\frac{q^2}{4p}\right)$$

where

$$\gamma_2 = \frac{i\ell'^2/a'^2}{[4(L'-x)/a'^2]-i} \tag{210}$$

(γ_2^* is the complex conjugate of γ_2).

We continue the calculation of the integral I_1. Setting $\gamma_1 = \gamma_1' + i\gamma_1''$, we find

$$\gamma_1' = -\frac{\ell'^2/a'^2}{1+\frac{4\xi^2}{a'^4}} \quad , \quad \gamma_1'' = \frac{(\ell'^2/a'^2)(2\xi/a'^2)}{1+\frac{4\xi^2}{a'^4}} \; .$$

Eq. (203a) can be transformed into

$$I_1 = L' \int_{-\infty}^{+\infty} \exp\left[-\frac{\xi^2}{a'^2}\right] \exp[\gamma_1'] \; \frac{\cos\gamma_1'' + \frac{2\xi}{a'^2}\sin\gamma_1''}{1+\frac{4\xi^2}{a'^4}} \, d\xi . \tag{211}$$

The first factor in the integrand is appreciable only for values of ξ which do not exceed a' in order of magnitude. In this region the term $2\xi/a'^2$ can be neglected as compared with unity, since $a' \gg 1$. Thus, assuming that $a' \gg 1$, the integral (211) becomes

$$I_1 = L' \exp\left[-\frac{\ell'^2}{a'^2}\right] \int_{-\infty}^{+\infty} \cos\left(\frac{\ell'^2}{a'^2} \; \frac{2\xi}{a'^2}\right) \exp\left[-\frac{\xi^2}{a'^2}\right] d\xi \quad , \tag{212}$$

and finally

$$I_1 = \sqrt{\pi}\, Lak^2 \exp\left[-\frac{\ell^2}{a^2}\right] \exp\left[-\left(\frac{\ell^2}{a^2} \; \frac{1}{ka}\right)^2\right] . \tag{213}$$

As a result of the change of variable $y = 4(L'-x)/a'^2$, the integral I_2 takes the form

$$I_2 = \frac{\sqrt{\pi}}{8i}\, a'^3 \int_0^D \left\{ \frac{\exp[(i\ell^2/a^2)/(y-i)]}{y - i} - \frac{\exp[-(i\ell^2/a^2)/(y+i)]}{y + i} \right\} dy . \tag{214}$$

If we introduce the new variable $z = (i\ell^2/a^2)/(y-i)$, then the first integral in (214) becomes

$$\int_0^D \frac{\exp[(i\ell^2/a^2)/(y-i)]}{y - i}\, dy = \int_{(i\ell^2/a^2)/(D-i)}^{-\ell^2/a^2} \frac{e^z}{z}\, dz.$$

After the change of variable $z = (-i\ell^2/a^2)/(y+i)$ the second integral in (214) becomes

$$\int_0^D \frac{\exp[-(i\ell^2/a^2)/(y+i)]}{y + i}\, dy = - \int_{-\ell^2/a^2}^{-(i\ell^2/a^2)/(D+i)} \frac{e^z}{z}\, dz.$$

Therefore the integral I_2 can be represented in the following form:

$$I_2 = \frac{\sqrt{\pi}}{8i} a'^3 [Ei(\epsilon_2) - Ei(\epsilon_1)], \tag{215}$$

where

$$\epsilon_2 = - \frac{i\ell^2/a^2}{D + i}, \quad \epsilon_1 = \frac{i\ell^2/a^2}{D - i}, \tag{216}$$

and Ei denotes the integral exponential function.

First of all we shall show that our formulas give the familiar expressions for the mean square amplitude and phase fluctuations in the special case $\ell = 0$. In this case, we find from (213) that the integral I_1 is

$$I_1 = \sqrt{\pi}\, k^2 aL. \tag{217}$$

Turning to the integral I_2, we use the following series representation of the integral exponential function Ei [39]:

$$Ei(x) = C + \log x + \sum_{k=1}^{k=\infty} \frac{x^k}{k \cdot k!} .$$

Then we obtain

$$Ei(\epsilon_2) - Ei(\epsilon_1) = \log \frac{\epsilon_2}{\epsilon_1} + \sum_{k=1}^{\infty} \frac{\epsilon_2^k}{k \cdot k!} - \sum_{k=1}^{\infty} \frac{\epsilon_1^k}{k \cdot k!} .$$

For $\ell = 0$ all terms of the series vanish and we obtain

$$\lim_{\ell \to 0} \left[\mathrm{Ei}(\epsilon_2) - \mathrm{Ei}(\epsilon_1)\right] = \log\left[-\frac{D - i}{D + i}\right] = 2i \text{ arc tan } D,$$

and

$$I_2 = \frac{\sqrt{\pi}}{4} k^3 a^3 \text{ arc tan } D. \tag{218}$$

Finally, using (217) and (218), we obtain the familiar formulas

$$\overline{S^2} = \frac{\sqrt{\pi}}{2} \overline{\mu^2} k^2 aL(1 + \frac{1}{D} \text{ arc tan } D), \tag{219}$$

$$\overline{B^2} = \frac{\sqrt{\pi}}{2} \overline{\mu^2} k^2 aL(1 - \frac{1}{D} \text{ arc tan } D). \tag{220}$$

Returning to the question of the transverse autocorrelation of the amplitude (and phase) fluctuations, we investigate the corresponding amplitude autocorrelation coefficient R_b and phase autocorrelation coefficient R_s, beginning with their definitions:

$$R_b = \frac{\overline{B_1 B_2}}{\overline{B^2}}, \tag{221}$$

$$R_s = \frac{\overline{S_1 S_2}}{\overline{S^2}}. \tag{222}$$

Using Eqs. (196), (197), (213) and (215), we obtain

$$R_b = \frac{\exp\left[-\frac{\ell^2}{a^2}\right] \exp\left[-\left(\frac{\ell^2}{a^2} \; \frac{1}{ka}\right)^2\right] - \frac{1}{2iD}\left[\mathrm{Ei}(\epsilon_2) - \mathrm{Ei}(\epsilon_1)\right]}{1 - \frac{1}{D} \text{ arc tan } D}, \tag{223}$$

$$R_s = \frac{\exp\left[-\frac{\ell^2}{a^2}\right] \exp\left[-\left(\frac{\ell^2}{a^2} \; \frac{1}{ka}\right)^2\right] + \frac{1}{2iD}\left[\mathrm{Ei}(\epsilon_2) - \mathrm{Ei}(\epsilon_1)\right]}{1 + \frac{1}{D} \text{ arc tan } D}. \tag{224}$$

As is apparent from these formulas, the autocorrelation coefficients depend only on the three dimensionless parameters ℓ/a, D, ka, where the dependence on the parameter ka can be neglected (in the case being considered $ka \gg 1$) by setting

$$\exp\left[-\left(\frac{\ell^2}{a^2}\ \frac{1}{ka}\right)^2\right] \sim 1.$$

Eqs. (223) and (224) then become

$$R_b = \frac{\exp\left[\frac{-\ell^2}{a^2}\right] - \frac{1}{2iD}\left[Ei(\epsilon_2) - Ei(\epsilon_1)\right]}{1 - \frac{1}{D} \text{ arc tan } D}, \tag{225}$$

$$R_s = \frac{\exp\left[\frac{-\ell^2}{a^2}\right] + \frac{1}{2iD}\left[Ei(\epsilon_2) - Ei(\epsilon_1)\right]}{1 + \frac{1}{D} \text{ arc tan } D}. \tag{226}$$

In the case of small values of the wave parameter ($D \ll 1$), these formulas can be simplified. In fact, from (216) we get the equality

$$\epsilon_2 - \epsilon_1 = \frac{-2i\ \frac{\ell^2}{a^2}\ D}{1 + D^2},$$

from which we see that the difference $\epsilon_2 - \epsilon_1$ is small provided that $\ell^2 D/a^2 \ll 1$. The last inequality will be met, for example, under the condition that $D \ll 1$ and $\ell \sim a$. If it is met, then we can confine ourselves to the leading terms in the expansion of $Ei(\epsilon_2) - Ei(\epsilon_1)$ in powers of small increments of the argument $\epsilon_2 - \epsilon_1$. The expansion procedure is simplified if we introduce a third point $\epsilon_o = -\ell^2/a^2$ lying halfway between the points ϵ_1 and ϵ_2 and if we represent the difference of the integral exponential functions in the form

$$\left[Ei(\epsilon_2) - Ei(\epsilon_o)\right] - \left[Ei(\epsilon_1) - Ei(\epsilon_o)\right].$$

The quantity in the second set of square brackets is the complex conjugate of the quantity in the first. Therefore, if we expand one of these quantities in series, we automatically

obtain the expansion of the other at the same time. We expand the first bracketed term in series about the point ϵ_o, i.e.

$$Ei(\epsilon_2) - Ei(\epsilon_o) = Ei'(\epsilon_o)\,\Delta\epsilon + \frac{1}{2}\,Ei''(\epsilon_o)\,\Delta\epsilon^2 + \frac{1}{6}\,Ei'''(\epsilon_o)\,\Delta\epsilon^3 + \ldots .$$

Here

$$Ei'(\epsilon_o) = \frac{e^{\epsilon_o}}{\epsilon_o}, \quad Ei''(\epsilon_o) = \left(\frac{1}{\epsilon_o} - \frac{1}{\epsilon_o^2}\right) e^{\epsilon_o},$$

$$Ei'''(\epsilon_o) = \left(\frac{1}{\epsilon_o} - \frac{2}{\epsilon_o^2} + \frac{2}{\epsilon_o^3}\right) e^{\epsilon_o},$$

$$\Delta\epsilon = \epsilon_2 - \epsilon_o = \frac{-\frac{\ell^2}{a^2} - i\,\frac{\ell^2}{a^2}\,D}{1 + D^2} + \frac{\ell^2}{a^2} \sim \epsilon_o(iD - D^2 - iD^3).$$

The expression for $\Delta\epsilon$ is given up to terms of the third power in D. With the same degree of accuracy, we obtain

$$Ei(\epsilon_2) - Ei(\epsilon_o) = e^{\epsilon_o}\left[iD - \frac{1}{2}(1 + \epsilon_o)D^2 - \frac{1}{3}(1 + 2\epsilon_o + \frac{1}{2}\,\epsilon_o^2)iD^3\right],$$

whence

$$Ei(\epsilon_2) - Ei(\epsilon_1) = 2iD\,e^{\epsilon_o}\left[1 - \frac{D^2}{3}(1 + 2\epsilon_o + \frac{1}{2}\,\epsilon_o^2\right].$$

In calculating the amplitude correlation coefficient R_b using Eq. (225), it is necessary to consider the quadratic term in D, since the leading terms cancel each other. In calculating the phase correlation coefficient R_s using Eq. (226), the quadratic term in D can be neglected. Correspondingly, writing

$$1 - \frac{1}{D}\,\text{arc tan}\,D \sim \frac{D^2}{3} \quad \text{and} \quad 1 + \frac{1}{D}\,\text{arc tan}\,D \sim 2,$$

we obtain

$$R_b = \exp\left(\frac{-\ell^2}{a^2}\right)\left[1 - 2\,\frac{\ell^2}{a^2} + \frac{1}{2}\left(\frac{\ell^2}{a^2}\right)^2\right], \tag{227}$$

$$R_s = \exp\left(\frac{-\ell^2}{a^2}\right) . \tag{227a}$$

The phase autocorrelation coefficient R_s has a Gaussian shape and coincides with the correlation coefficient for the refractive index fluctuations. The amplitude autocorrelation coefficient R_b is not Gaussian. However, in this case also, as can be seen from Fig. 5, the correlation between the amplitude fluctuations extends over a distance of the same order as the correlation between the fluctuations of the refractive index.

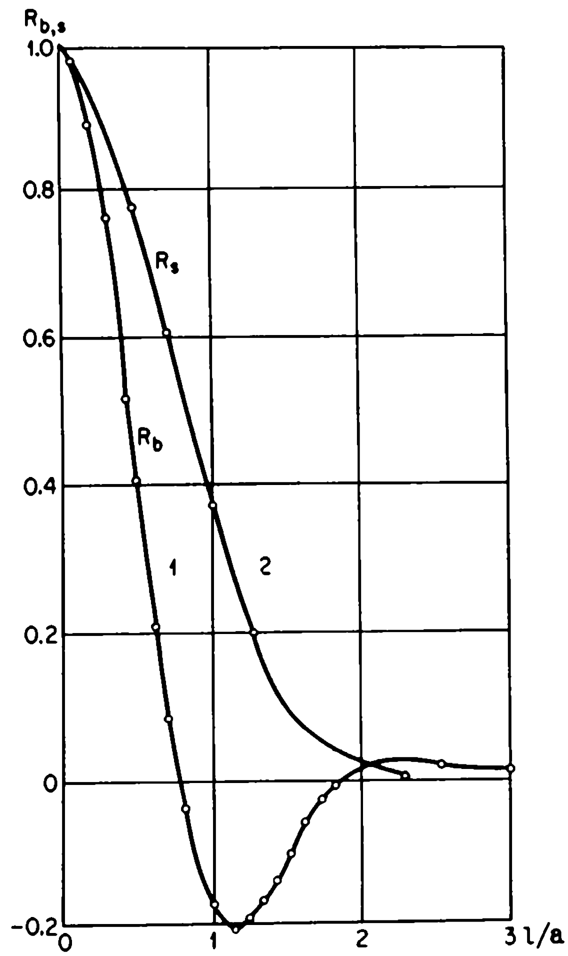

Fig. 5 The transverse autocorrelation for the case of small D (D << 1):
1 - the autocorrelation coefficient of the amplitude fluctuations;
2 - the autocorrelation coefficient of the phase fluctuations, identical with the autocorrelation coefficient of the refractive index fluctuations.

In the region of large values of the wave parameter $(D \gg 1)$

$$\epsilon_2 \sim -\frac{i}{D}\frac{\ell^2}{a^2}, \quad \epsilon_1 \sim \frac{i}{D}\frac{\ell^2}{a^2}.$$

Using the formula

$$\mathrm{Ei}(-ix) - \mathrm{Ei}(ix) = 2i\frac{\pi}{2} - 2i\mathrm{Si}x,$$

we transform the autocorrelation coefficients (225) and (226) into the form

$$R_b = \frac{\exp\left[-\frac{\ell^2}{a^2}\right] - \frac{1}{D}\left[\frac{\pi}{2} - \mathrm{Si}\left(\frac{1}{D}\cdot\frac{\ell^2}{a^2}\right)\right]}{1 - \frac{1}{D}\arctan D}, \tag{228}$$

$$R_s = \frac{\exp\left[-\frac{\ell^2}{a^2}\right] + \frac{1}{D}\left[\frac{\pi}{2} - \mathrm{Si}\left(\frac{1}{D}\cdot\frac{\ell^2}{a^2}\right)\right]}{1 + \frac{1}{D}\arctan D}. \tag{229}$$

The dependence of the autocorrelation coefficients on the distance ℓ between the receivers is shown graphically in Fig. 6 for the case $D = 10$. The middle curve shows the correlation coefficient $\exp(-\ell^2/a^2)$ of the refractive index. From the curves it is clear that the transverse autocorrelation of amplitude and phase fluctuations extends to a distance of the order of the correlation distance of the inhomogeneities in the medium. We came to the same conclusion in considering the case of small values of the wave parameter. Therefore, we can assume that this result also remains true in the region of intermediate values of the wave parameter $(D \sim 1)$.

Of course, it is more difficult to investigate the dependence of the autocorrelation coefficients R_b and R_s on the parameter ℓ/a in the intermediate region. The difficulty is connected with the fact that the integral exponential function is not tabulated for complex values of the argument. However, we can obtain the asymptotic values of the autocorrelation coefficients for sufficiently large values of the ratio ℓ/a if we use the asymptotic expression of the function Ei encountered in the preceding chapter. In fact, in this case

$$|\epsilon_2| = |\epsilon_1| = \frac{\ell^2/a^2}{\sqrt{1 + D^2}} \gg 1.$$

Restricting ourselves to the first term in the expansion (143), we obtain

$$\mathrm{Ei}(\epsilon_2) = - \frac{1}{\frac{\ell^2}{a^2}\frac{1 + iD}{1 + D^2}} \exp\left[- \frac{\ell^2}{a^2}\frac{1 + iD}{1 + D^2}\right] .$$

Here we neglected the remainder term R_1

$$|R_1| < \frac{1}{\left(\frac{\ell^2}{a^2}\frac{1}{\sqrt{1 + D^2}}\right)^2 \frac{\sqrt{2}}{2}}$$

as compared with the term considered, which is equal to

$$\frac{1}{\frac{\ell^2}{a^2}\frac{1}{\sqrt{1 + D^2}}}$$

in absolute value. After some elementary manipulation, we obtain

$$R_b \sim \frac{\exp(-\frac{\ell^2}{a^2}) - \left[\frac{a^2}{\ell^2 D}\sin(\frac{\ell^2}{a^2}\frac{D}{1+D^2}) + \frac{a^2}{\ell^2}\cos(\frac{\ell^2}{a^2}\frac{D}{1+D^2})\right]\exp(-\frac{\ell^2}{a^2}\frac{1}{1+D^2})}{1 - \frac{1}{D}\arctan D} , \qquad (230)$$

$$R_s \sim \frac{\exp(-\frac{\ell^2}{a^2}) + \left[\frac{a^2}{\ell^2 D}\sin(\frac{\ell^2}{a^2}\,\frac{D}{1+D^2}) + \frac{a^2}{\ell^2}\cos(\frac{\ell^2}{a^2}\,\frac{D}{1+D^2})\right]\exp(-\frac{\ell^2}{a^2}\,\frac{1}{1+D^2})}{1+\frac{1}{D}\arctan D} \qquad (231)$$

for the autocorrelation coefficients. Eqs. (230) and (231) contain only squares of the ratio ℓ/a, which makes them applicable for ℓ/a as small as 3. For such a value of the ratio ℓ/a and for $D \sim 1$, the autocorrelation coefficients are small as compared with unity. Consequently, in the intermediate range, the autocorrelation of amplitude (and phase) fluctuations also extends over a distance of the order of the correlation distance of the inhomogeneities in the medium.

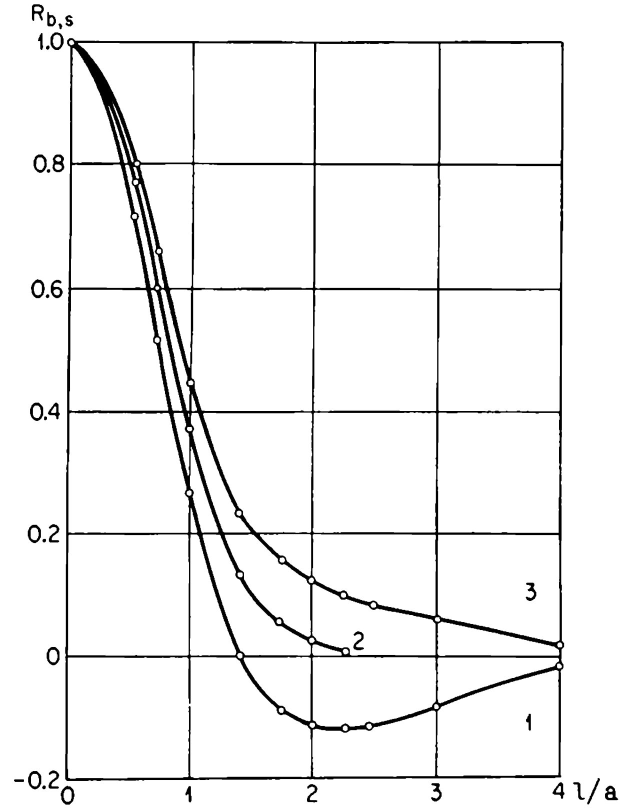

Fig. 6 The transverse autocorrelation for the case of large D (D = 10): 1 - the autocorrelation coefficient of the amplitude fluctuations; 2 - the autocorrelation coefficient of the refractive index fluctuations; 3 - the autocorrelation coefficient of the phase fluctuations.

26. The Quasi-Static Condition. Until now we have regarded the distribution of inhomogeneities as static, neglecting their change as a result of heat conduction, diffusion, convection and drift. We can neglect this change if the propagation time $t = L/c$ is small compared to the characteristic time scale of change in the inhomogeneities. However, if this condition is not met, then in calculating the amplitude (or phase) fluctuations at the time t, we have to take into account the refractive index fluctuations at the time $t' = t - r/c$, where r is the distance from the scattering element to the observation point. In this case, the basic formulas (106) and (107) are written as follows:

$$S(L',0,0,t) = \int_0^{L'}\int\int_{-\infty}^{+\infty} \Phi_1(L'-\xi',\rho')\,\mu(\xi',\eta',\zeta',t')\,d\xi' d\eta' d\zeta', \tag{232}$$

$$B(L',0,0,t) = \int_0^{L'}\int\int_{-\infty}^{+\infty} \Phi_2(L'-\xi',\rho')\,\mu(\xi',\eta',\zeta',t')\,d\xi' d\eta' d\zeta', \tag{233}$$

where $t' = t - r/c$. Squaring and averaging (232) and (233), we obtain

$$\overline{S^2} = \overline{\mu^2} \int_0^{L'}\int_0^{L'}\int\int\int\int_{-\infty}^{+\infty} \Phi_1(L'-\xi_1', \rho_1')\,\Phi_1(L'-\xi_2', \rho_2') \times \tag{234}$$

$$\times N(\xi_1' - \xi_2',\eta_1' - \eta_2',\zeta_1' - \zeta_2',t' - t'')\,d\xi_1' d\xi_2', d\eta_1' d\eta_2' d\zeta_1' d\zeta_2' ,$$

$$\overline{B^2} = \overline{\mu^2} \int_0^{L'}\int_0^{L'}\int\int\int\int_{-\infty}^{+\infty} \Phi_2(L'-\xi_1', \rho_1')\,\Phi_2(L'-\xi_2', \rho_2') \times \tag{235}$$

$$\times N(\xi_1' - \xi_2',\eta_1' - \eta_2',\zeta_1' - \zeta_2',t' - t'')\,d\xi_1' d\xi_2' d\eta_1' d\eta_2' d\zeta_1' d\zeta_2',$$

where

$$N(\xi_1' - \xi_2',\ \eta_1' - \eta_2',\ \zeta_1' - \zeta_2',\ t' - t'') = \frac{1}{\overline{\mu^2}}\,\overline{\mu(\xi_1',\eta_1',\zeta_1',t')\mu(\xi_2',\eta_2',\zeta_2',t'')}\ .$$

Here we assume that the correlation coefficient N depends only on the coordinate and time differences, i.e., we assume that the process is stationary in time and homogeneous in space. The time difference is

$$t' - t'' = (t - r_1/c) - (t - r_2/c) = (r_2 - r_1)/c. \tag{236}$$

The correlation coefficient N is different from zero if $r_1 - r_2$ does not exceed the correlation distance a in order of magnitude. This means that the time difference $t' - t''$ does not exceed the quantity a/c in order of magnitude, i.e.

$$t' - t'' \sim a/c \,. \tag{237}$$

The quantity a/c defines the time it takes the wave to go a distance equal to the correlation distance (the scale of the inhomogeneities). If this time is small compared to the correlation time T of the refractive index, then the time difference $t' - t''$ can be set equal to zero, and any explicit time dependence in our formulas vanishes. Consequently, the quasi-static condition takes the form of the inequality

$$a/c \ll T. \tag{238}$$

The quasi-static condition (238) for averaged quantities is very much less stringent than the quasi-static condition for unaveraged quantities, which clearly can be written as

$$L/c \ll T.$$

In all actual cases, the quasi-static condition (238) for averaged quantities is evidently met with a big margin. This justifies the quasi-static assumption which we have made from the very beginning.

<u>27. The Time Autocorrelation of the Amplitude Fluctuations.</u> The change of the inhomogeneities in time produces a change in the frequency of the scattered waves and broadens the

frequency bandwidth of the incident radiation. The nature of this broadening can be inferred from the form of the time autocorrelation functions of the amplitude and phase fluctuations. Thus, for example, the frequency bandwidth is determined by the correlation time of the radiation. However, we shall not be concerned with the frequency structure of the fluctuations, and only the autocorrelation functions are examined below.

An appreciable weakening of the autocorrelation of the amplitude fluctuations can be expected after a time interval τ commensurate with the correlation time T of the refractive index. If the quasi-static condition (238) is met, then the time interval τ is also large compared to the time a/c, i.e.

$$\tau \gg a/c. \tag{239}$$

We write Eq. (233) for the times t_1 and t_2, assuming that they are separated by the interval τ:

$$B(L',0,0,t_1) = \int_0^{L'} \iint_{-\infty}^{+\infty} \Phi_2(L' - \xi', \rho')\, \mu(\xi', \eta', \zeta', t')d\xi'd\eta'd\zeta', \tag{240}$$

$$B(L',0,0,t_2) = \int_0^{L'} \iint_{-\infty}^{+\infty} \Phi_2(L' - \xi', \rho')\, \mu(\xi', \eta', \zeta', t'')d\xi'd\eta'd\zeta', \tag{241}$$

where

$$t' = t_1 - r/c \ , \qquad t'' = t_2 - r/c \ . \tag{242}$$

Denoting the time autocorrelation function of the amplitude by F(t), by definition we have

$$F(\tau) = \overline{B(L',0,0,t_1)\ B(L',0,0,t_2)} \ .$$

Multiplying (240) and (241) and averaging, we obtain

$$F(\tau) = \overline{\mu^2} \int_0^{L'}\int_0^{L'} \iiiint_{-\infty}^{+\infty} \Phi_2(L' - \xi_1', \rho_1')\ \Phi_2(L' - \xi_2', \rho_2') \times \tag{243}$$

$$\times N(\xi_1' - \xi_2', \eta_1' - \eta_2', \zeta_1' - \zeta_2', t' - t'')d\xi_1'd\xi_2'd\eta_1'd\eta_2'd\zeta_1'd\zeta_2' \ .$$

By (242) we have

$$t' - t'' = (t_1 - r_1/c) - (t_2 - r_2/c) \tag{244}$$

$$= t_1 - t_2 + (r_2 - r_1)/c = \tau + (r_2 - r_1)/c \sim \tau ,$$

since $(r_2 - r_1)/c \sim a/c$. Therefore Eq. (243) becomes

$$F(\tau) = \overline{\mu^2} \int_0^{L'} \int_0^{L'} \int\int\int_{-\infty}^{+\infty}\int \Phi_2(L' - \xi_1', \rho_1') \Phi_2(L' - \xi_2', \rho_2') \times \tag{245}$$

$$\times N(r',\tau) d\xi_1' d\xi_2' d\eta_1' d\eta_2' d\zeta_1' d\zeta_2' .$$

Mintzer [23] started with the assumption that the correlation coefficient $N(r',t)$ separates into two factors, one of which depends only on the coordinates and the other only on the time, i.e., that the correlation coefficient can be represented in the form

$$N(r') M(\tau). \tag{246}$$

Then (245) becomes

$$F(\tau) = M(\tau) \overline{\mu^2} \int_0^{L'} \int_0^{L'} \int\int\int_{-\infty}^{+\infty}\int \Phi_2(L' - \xi_1', \rho_1') \Phi_2(L' - \xi_2', \rho_2') \times \tag{247}$$

$$\times N(r') d\xi_1' d\xi_2' d\eta_1' d\eta_2' d\zeta_1' d\zeta_2' .$$

Denoting the amplitude autocorrelation coefficient by $R(\tau)$, we have

$$R(\tau) = \frac{F(\tau)}{\overline{B^2}} \tag{248}$$

by definition. Then by (247) and (109) we get the very simple result

$$R(\tau) = M(\tau) . \tag{249}$$

Thus $R(\tau)$, the time autocorrelation coefficient of the amplitude, coincides with $M(\tau)$, the time autocorrelation coefficient of the refractive index. It is possible that Mintzer's assumption that the time and space coordinates can be separated is justified in the absence of any motion of the inhomogeneities which are produced by drift and convection. In this case, change of the inhomogeneities in time might be produced, for example, by turbulence, heat conduction and diffusion. However, such a separation cannot be valid when drift or convection is present. In this regard, it is of interest to examine the case where the change in the inhomogeneities in time is caused by their motion.

We assume that all the inhomogeneities move with the same velocity $\vec{v}$, as a result of drift, say, and we assume that the change in the inhomogeneities results exclusively from the drift, while other factors (turbulence, heat conduction, diffusion) play no important role, i.e., changes produced by these factors proceed much more slowly. Then in the coordinate system moving with the flow, the correlation coefficient depends only on the coordinates, i.e., has the form

$$N(x_1 - x_2, y_1 - y_2, z_1 - z_2).$$

Using the Galilean transformation formulas

$$\begin{aligned} x_1 - x_2 &= \xi_1 - \xi_2 - v_x\tau, \\ y_1 - y_2 &= \eta_1 - \eta_2, \\ z_1 - z_2 &= \zeta_1 - \zeta_2 - v_z\tau, \end{aligned} \tag{250}$$

we obtain

$$N(\xi_1 - \xi_2 - v_x\tau , \eta_1 - \eta_2, \zeta_1 - \zeta_2 - v_z\tau) \tag{251}$$

in the coordinate system ξ,η,ζ fixed with respect to the receiver. (The coordinate systems are oriented in such a way that the flow velocity $\vec{v}$ lies in the planes xoz and $\xi o\zeta$.) It can be seen from (251) that in the case under consideration there is no separation of the space and time variables.

It is easy to see that in the case of a homogeneous flow the problem of time correlation reduces to the problem of space correlation which has already been solved. To see this it suffices to use the principle of relativity to go over to the coordinate system x,y,z moving with the flow. The receiver moves with the velocity $-\vec{v}$ with respect to this system. We assume that at time t the receiver is located at the point A and that at time $t + \tau$ it is located at the point E, where the receiver displacement $AE = \vec{\ell}$ satisfies the condition $\vec{\ell} = \vec{v}\tau$. We denote the amplitude fluctuation at the point A at the time t by B(A,t) and the amplitude fluctuation at the point E at the time $t + \tau$ by $B(E,t + \tau)$. Since the distribution of inhomogeneities can be considered static in the coordinate system moving with the flow, inasmuch as we are dealing with time intervals of order τ, the amplitude at any point does not change with time, i.e.

$$B(E,t + \tau) = B(E,t).$$

Therefore

$$\overline{B(A,t)\ B(E,t + \tau)} = \overline{B(A,t)\ B(E,t)}\ . \tag{252}$$

Here the left hand side represents the time correlation function $F(\tau)$ in the coordinate system fixed with respect to the observer, while the right hand side represents the space correlation function $F_1(\vec{\ell})$. Thus we have

$$F(\tau) = F_1(\vec{\ell}). \tag{253}$$

Since the longitudinal displacement in a time τ of order a/v cannot produce an appreciable decorrelation, in Eq. (253) it is enough to consider only the transverse displacement $\ell_z = v_z\tau$. Then we get

$$F(\tau) = F_1(\ell_z). \tag{254}$$

We note that the correlation functions of the phase fluctuations also satisfy the relation (254). It only remains to use the corresponding results of Section 26, where the transverse displacement was denoted by ℓ. Thus, for example, Eqs. (225) and (226) give

$$R_b(\tau) = \frac{\exp\left[-\frac{v_z^2}{a^2}\tau^2\right] - \frac{1}{2iD}\left[Ei(\epsilon_2) - Ei(\epsilon_1)\right]}{1 - \frac{1}{D}\arctan D}, \tag{255}$$

$$R_s(\tau) = \frac{\exp\left[-\frac{v_z^2}{a^2}\tau^2\right] + \frac{1}{2iD}\left[Ei(\epsilon_2) - Ei(\epsilon_1)\right]}{1 + \frac{1}{D}\arctan D}, \tag{256}$$

where

$$\epsilon_2 = -\frac{i\frac{v_z^2}{a^2}\tau^2}{D + i}, \quad \epsilon_1 = \epsilon_2^*. \tag{257}$$

In the Fraunhofer zone ($D \gg 1$), they take the particularly simple form

$$R_b(\tau) = R_s(\tau) \sim \exp\left[-\frac{v_z^2}{a^2}\tau^2\right]. \tag{258}$$

Defining

$$T = \frac{a}{v_z}, \tag{259}$$

we get

$$R_b(\tau) = R_s(\tau) = \exp\left(-\frac{\tau^2}{T^2}\right). \tag{260}$$

It is clear from (259) that the correlation time T depends on the transverse component of the flow velocity or (what amounts to the same thing) on the transverse component of the receiver's velocity.

28. Comparison with Experiment. This section is devoted to discussing some acoustical and optical phenomena from the standpoint of the fluctuation theory presented in the two preceding chapters and then comparing the theory with experiment. The first experimental studies of amplitude and phase fluctuations when sound is propagated in the atmosphere are due to Krasilnikov [52,53]. Later, using an improved method, Krasilnikov and Ivanov-Shyts [7,8]

also made experimental studies of amplitude and phase fluctuations when sound waves are propagated in the atmosphere, and they compared the experimental results with the theoretical results obtained by Krasilnikov [6] using the ray approach. They discovered that the theoretical result concerning the phase fluctuations was in satisfactory agreement with experiment, while the theoretical formula for the amplitude fluctuations was not supported by the experimental data. Fig. 7 shows a logarithmic plot of the dependence of the mean square phase fluctuations (over various time intervals) on the distance between the transmitter and receiver.

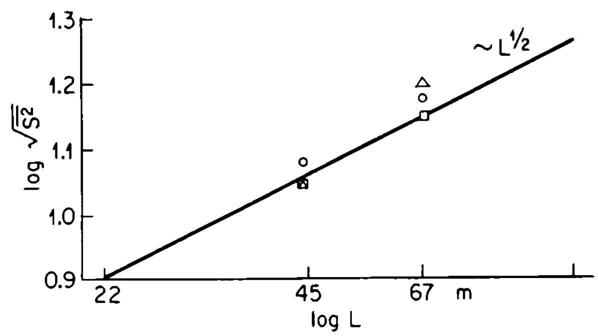

Fig. 7 Dependence of $\overline{S^2}$ on L for the frequency $\nu = 3$ Kcps (logarithmic scales): $\triangle, \Delta t = 0.04$ sec; $\circ, \Delta t = 0.08$ sec; $\square$ $, \Delta t = 0.2$ sec.
(After Krasilnikov and Ivanov-Shyts)

The line $L^{1/2}$ is drawn through the experimental point corresponding to the distance L = 22 m. It is clear from the figure that the theoretical law, according to which $\sqrt{\overline{S^2}} \sim L^{1/2}$, agrees satisfactorily with the experimental data. The dependence of the mean square amplitude fluctuation $\sqrt{\overline{B^2}}$ on the distance L is shown in Fig. 8.

The location of the experimental points with respect to the curves $L^{1/2}, L, L^{3/2}$ gives an idea of the behavior of $\sqrt{\overline{B^2}}$ as the distance changes. As can be seen from the figure, the amplitude fluctuations, like the phase fluctuations, grow with distance in proportion to $L^{1/2}$, whereas the ray theory leads to the stronger dependence $\sqrt{\overline{B^2}} \sim L^{3/2}$ of the amplitude fluctuations on distance. Trying to explain the discrepancy between theory and experiment, Krasilnikov [7] (and others) proposed that the ray approach is not suitable under the conditions of the experiment. Apparently, the cause for the discrepancy was correctly ascribed. In fact,

the condition that the wavelength be small compared to the scale of the inhomogeneities is not met: The inner dimension of the turbulence in the layer of the atmosphere near the earth is of the order of 1 cm, which is much smaller than the wavelength of the wave used in the experiment, namely 6.6 cm (5000 Kcps). On the other hand, the diffraction theory gives the dependence of the amplitude fluctuations on distance which is experimentally observed, i.e. $\sqrt{\overline{B^2}} \sim L^{1/2}$.

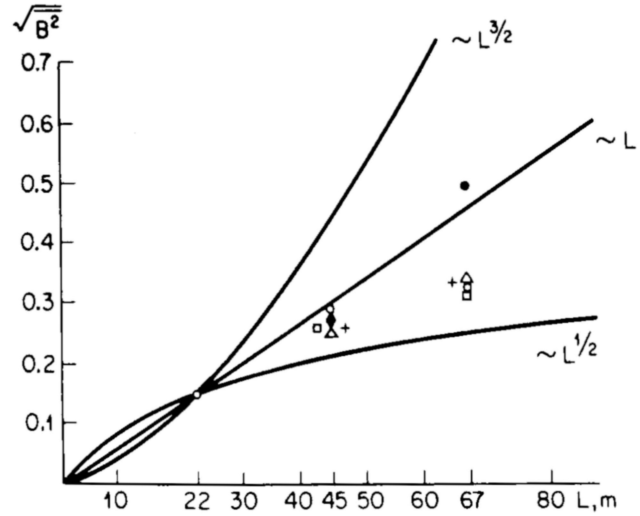

Fig. 8 Dependence of $\sqrt{\overline{B^2}}$ on L: o, measurements of Aug. 1, frequency $\nu = 5$ Kcps; △, July 31, $\nu = 5$ Kcps; +, Aug. 31, $\nu = 3$ Kcps; •, Aug. 9, $\nu = 3$ Kcps; □, Aug. 10, $\nu = 3$ Kcps. (After Krasilnikov and Ivanov-Shyts)

Sheehy [15] studied the fluctuations of signals propagated in deep water. In his experiments a series of sound pulses was sent at each fixed distance and the size of the relative fluctuations of the pressure amplitude was determined and expressed as a percentage of the mean amplitude (see Fig. 9). The experimental line (1 in Fig. 9) shows that the size of the fluctuations grows as the square root of the distance, although the scatter of the experimental points is large. Using Liebermann's data ($\overline{\mu^2} = 5 \times 10^{-9}$, a = 60 cm) and a correlation coefficient of the form e^{-r^2/a^2}, Mintzer [21] represented on the same graph the dependence

$$V = \sqrt{\pi^{1/2}\, \overline{\mu^2}\, k^2 aL}$$

between the relative pressure fluctuation V and the distance L (line 3) and also the dependence $V \sim L^{3/2}$ which follows from the ray theory (line 2). As can be seen from the graph, the latter dependence is in strong contradiction with experiment. This is explained by the

fact that Sheehy's data pertain to the Fraunhofer zone. In fact, for distances in the range 50 m to 2500 m for a frequency ν = 24 Kcps and for a scale of inhomogeneities a = 60 cm, the wave parameter varies from 6 to 300, i.e., is much larger than 1. The results of the diffraction theory (line 3) are in good agreement with the experimental data. We note that between the dashed boundary lines, containing 90 percent of the data, the quantity $\overline{\mu^2}a$ varies from 1.5×10^{-6} to 1.5×10^{-7}. (The experimental line 1 corresponds to the value 5×10^{-7}.)

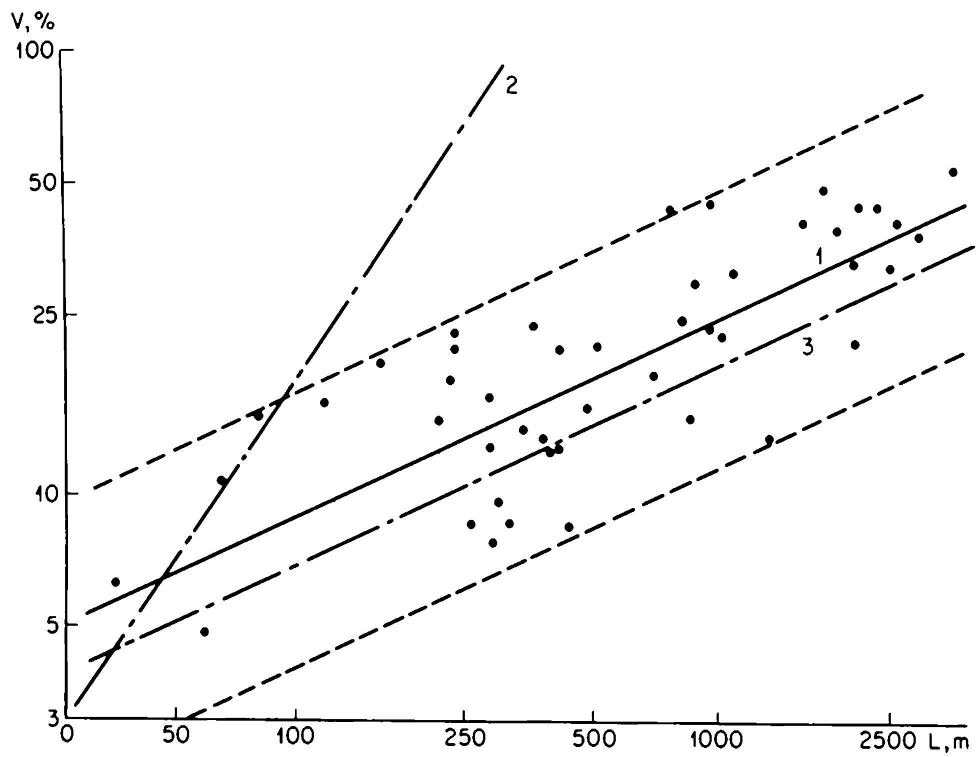

Fig. 9 Dependence of the size of the relative fluctuation of the pressure amplitude of the direct signal on distance. (After Sheehy and Mintzer)

In optics, because of the smallness of the wavelength, the wave parameter $D = 4L/ka^2$ can stay small compared to unity even over large distances; as a result the region of applicability of the ray theory is greatly extended. Convincing support for this can be found in the phenomenon of the twinkling of stars. Butler [16] studied experimentally the brightness fluctuations of stars as a function of the zenith distance. The results of his observations are shown by the points in Fig. 10. The path of the ray in the atmosphere is L = h sec z, where z is the zenith distance and h is the height of the layer containing the inhomogeneities.

Therefore in the graph the relative brightness fluctuation is given as a function of sec z.

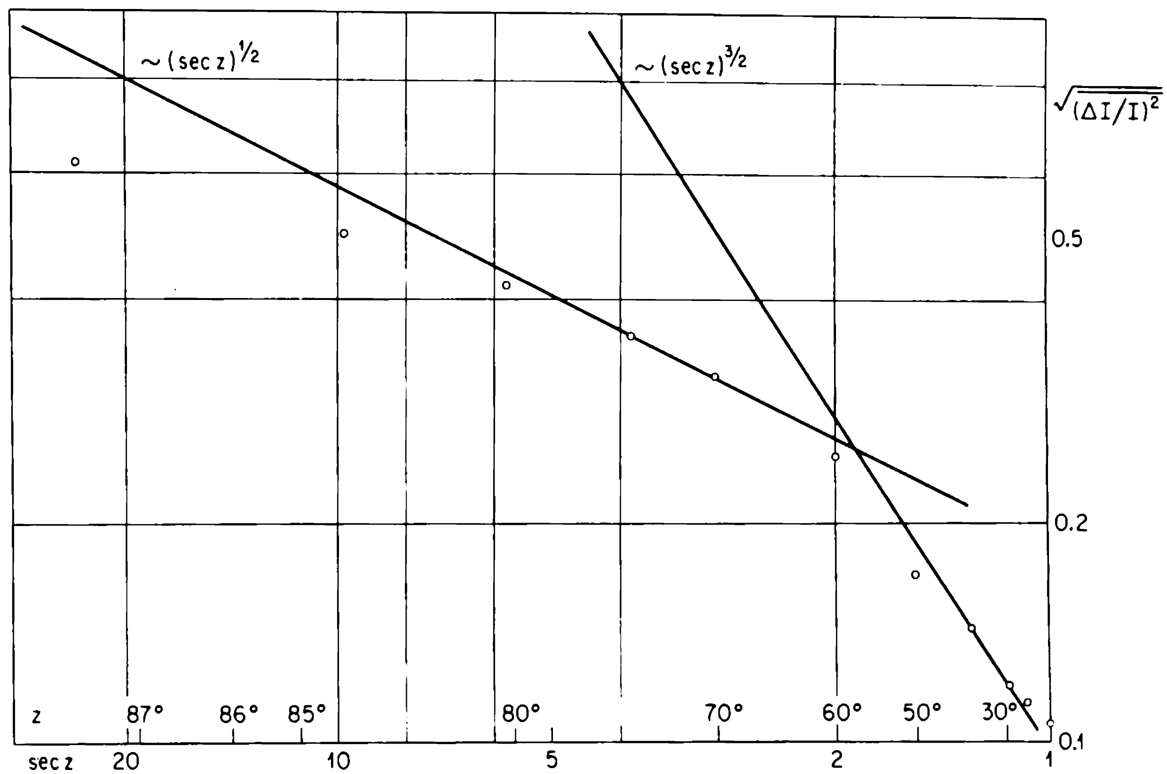

Fig. 10 Dependence of the relative brightness fluctuation of stars on the zenith angle. (After Butler and Megaw)

Megaw [17] compared these data with theory. As can be seen from the graph, the dependence $\sqrt{\overline{(\Delta I/I)^2}} \sim (\sec z)^{3/2}$, holds for zenith distances which do not exceed 60°. At greater zenith distances the brightness variation obeys the law $\sqrt{\overline{(\Delta I/I)^2}} \sim (\sec z)^{1/2}$. The graph is interesting in that it encompasses both laws at once and shows quite clearly the transition from one law to the other. Van Isacher [19] draws similar conclusions from Butler's observations.

Experimental material on the correlation of fluctuations is extraordinarily skimpy, and that which is available requires correct interpretation. Let us examine some experimental hydroacoustic data pertaining to time correlation of amplitude fluctuations. Such data (obtained by Sheehy) is presented in Mintzer's paper [23]. Analyzing this data, Mintzer came to the conclusion that the time correlation coefficient does not depend on the distance. On the other hand, experiments carried out at the same distance but at different times give widely differing curves for the correlation coefficient. Since the experiments were carried out on moving ships, it is natural to assume that the difference in the curves was produced

by the difference in the ships' motion. Starting with this assumption, we can interpret the results of the experiments, using the theory developed in the preceding section.

The experiment consisted of 50 series of pulses. Each series consisted of 50 pulses at a frequency of 24 Kcps. In 16 series the length of the pulses was 10 msec, and in the remaining series it was 30 msec. The transmitter and receiver were at a depth of 15 m. Since the pulses are separated by one second intervals, the values of the correlation coefficient were found for $\tau = 1,2,3,...$ sec. Fig. 11 shows the time $\tau^{(0)}$ of the first zero of the correlation function as a function of the distance between the transmitter and the receiver.

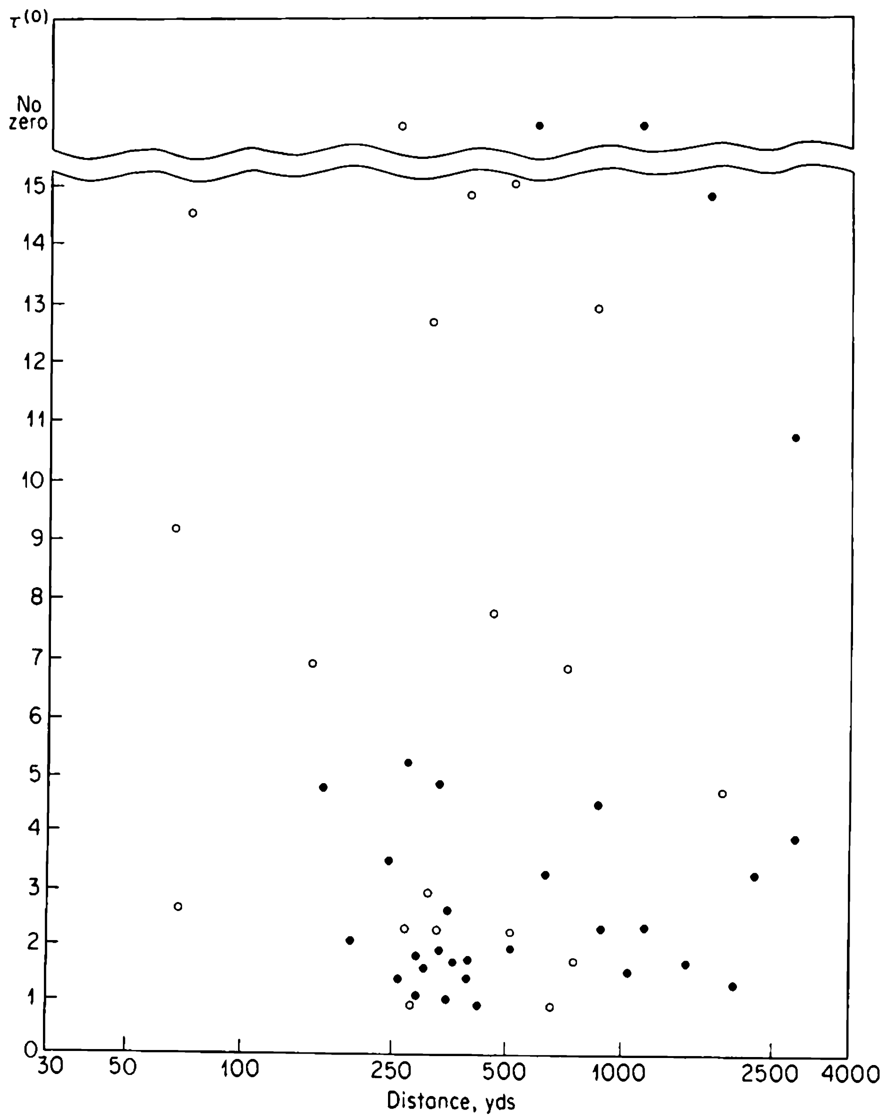

Fig. 11 Time $\tau^{(0)}$ of the first zero of the correlation function as a function of the distance. The circles correspond to pulses of length 10 msec, points correspond to pulses of length 30 msec. (After Sheehy and Mintzer)

The points and the circles refer to the pulses of lengths 30 msec and 10 msec respectively. It appears that there is no particular dependence of $\tau^{(0)}$ on distance. On the other hand, the variability of $\tau^{(0)}$ for different series at the same distance from transmitter to receiver is striking. Since there is no dependence on distance, the correlation function can be found by averaging over all 50 series. The values of the correlation coefficient obtained as a result of this averaging are denoted in Fig. 12 by circles.

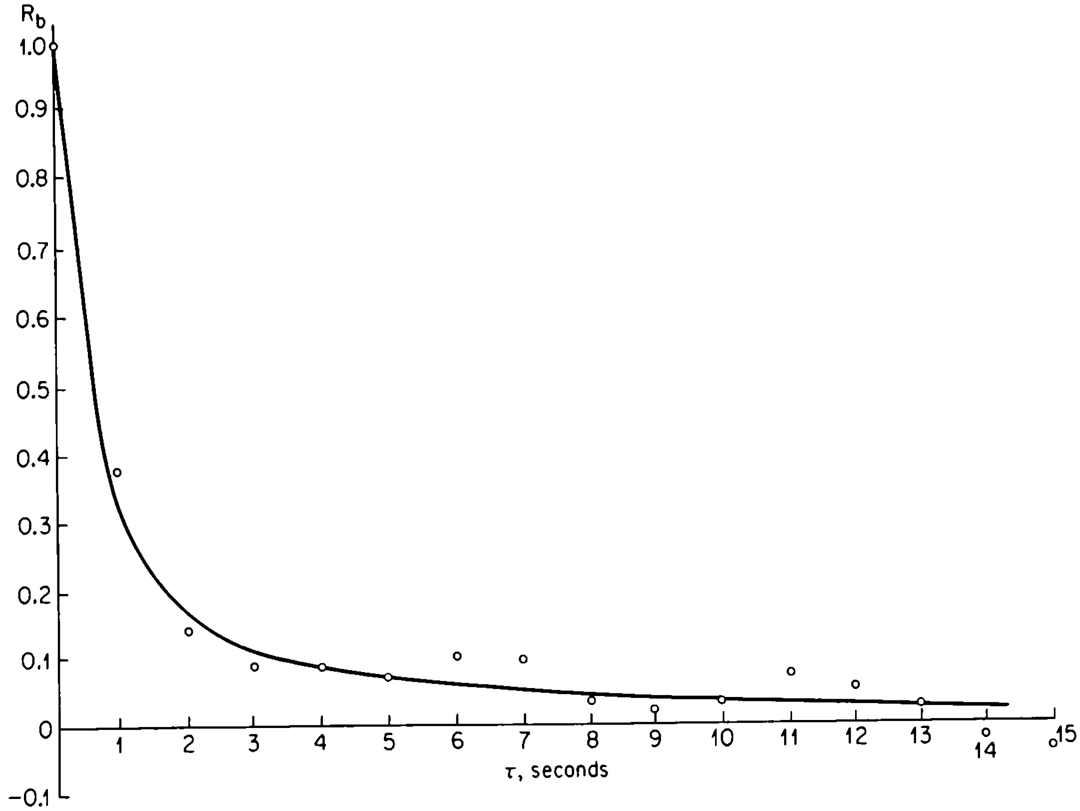

Fig. 12 Dependence of the average correlation coefficient R_b on time. The experimental points were obtained by averaging over 50 series of sound pulses. (After Sheehy and Mintzer)

The variability of the correlation coefficient can be explained by its strong dependence on the velocity of motion of the receiver, which is apparent from Eq. (258). The time dependence of the correlation coefficient shown in Fig. 12 can be explained theoretically, if we average the autocorrelation coefficient (258) over all the series, assuming that the ship's velocity with respect to the water changes from series to series in a random way and that the distribution of the ship's velocity obeys the normal law

$$W(v_z) = \frac{1}{\sqrt{2\pi}\, v} \exp\left[-\frac{v_z^2}{2v^2}\right] , \tag{261}$$

where v is the rms value of the component v_z. Using (261) to average (258), we obtain

$$R_b(\tau) = \int_{-\infty}^{+\infty} \exp\left[-\frac{v_z^2}{a^2}\tau^2\right] W(v_z)dv_z \tag{262}$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\, v} \exp\left[-\left(\frac{1}{2v^2} + \frac{\tau^2}{a^2}\right) v_z^2\right] dv_z = \frac{1}{\sqrt{1 + \frac{2v^2}{a^2}\tau^2}} ,$$

or

$$R_b(\tau) = \frac{1}{\sqrt{1 + \frac{\tau^2}{T^2}}} , \tag{263}$$

where

$$T = \frac{1}{\sqrt{2}} \frac{a}{v} .$$

If we set T = 0.33 sec, then the function (263) is represented by the curve in Fig. 12, which describes the behavior of the experimental points in a satisfactory way. If the scale of inhomogeneities is a = 60 cm, the mean velocity of the ship is $v \sim 128$ cm/sec, i.e. about 4.5 km/hour. Unfortunately, no indication of the ship's velocity is given in [23].

Part III

THE INFLUENCE OF FLUCTUATIONS ON THE DIFFRACTION IMAGE OF A FOCUSING SYSTEM

INTRODUCTORY REMARKS

Among the elements of a focusing system there is usually a part which converts a plane wave into a spherical wave; for example, this may be a lens or a mirror. Since the construction of the focusing part is not important, in what follows we shall suppose it to be a lens. The field near the focus cannot be calculated using ray theory, since ray theory gives an infinite intensity at the focus. By using wave considerations we can find the intensity distribution near the focus, the so-called diffraction image. Fluctuations of amplitude and phase in the incident wave give rise to fluctuations of the diffraction image, i.e. the image "quivers". When this happens not only do we observe deviations of the intensity from the mean distribution, but also the mean distribution itself depends in an essential way on the fluctuations in the incident wave. In this regard, two problems arise in the theory of focusing systems:

1) Finding the mean distribution in the diffraction image,

2) Finding the distribution of fluctuations in the diffraction image.

Neither of these problems, as far as we know, has been studied before. The problem of small fluctuations at the focus of an objective has been studied only in the paper of Krasilnikov and Tatarski [10]; their formulas can be obtained as a special case of our more general theory. In this book, we consider both questions without restricting ourselves by the requirement that the fluctuations be small*.

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* The basic material contained in Part III is given in the author's papers [59,60].

Chapter VII

GENERAL FORMULAS

29. The Debye Formula. If the distribution of amplitude $A(\vec{F})$ and phase $S(\vec{F})$ is known on the surface of a sphere behind the lens, then the field near the focus can be calculated by Debye's formula [42,50]:

$$p = \frac{i}{\lambda F} e^{-ikF} \int_S A(\vec{F})\, e^{iS(\vec{F})}\, e^{ik\vec{F}\cdot\vec{r}/F}\, ds. \tag{1}$$

Here the origin of the coordinate system is chosen at the focus (the center of the sphere). The location of the element ds of the spherical surface is given by the vector $\vec{F}$, and the location of the observation point is given by the vector $\vec{r}$. The physical meaning of Eq. (1) is clear. In fact, the distance R from the element ds of the sphere to the observation point is given by the following formula:

$$R = \sqrt{F^2 + r^2 - 2\vec{F}\cdot\vec{r}} \sim F - \vec{F}\cdot\vec{r}/F ,$$

if the distance r is much less than the focal distance ($r \ll F$). The quantity $k(F - \vec{F}\cdot\vec{r}/F)$ represents the optical path length from the element ds to the observation point. Therefore, the contribution of the element ds of the sphere to the field at the observation point is equal to

$$\frac{A(\vec{F})}{F}\; e^{iS(\vec{F})}\; e^{-ik(F-\vec{F}\cdot\vec{r}/F)}\; ds.$$

Integrating over the part of the sphere bounded by the diaphragm of the lens, we get a formula which differs from (1) only by the factor i/λ. This factor can be explained if we solve the problem more rigorously, using the Kirchoff formula. However, dimensional considerations also show the need for the factor $1/\lambda$.

Eq. (1) was obtained by Debye under the assumption that there is no phase aberration (S=0)

and that the amplitude of the spherical wave along the wave front is constant (A = const). The Debye formula remains valid when phase aberration is present and when the amplitude is not constant, provided that the change of the phase S and the relative change of the amplitude A in a distance of a wavelength along the surface of integration is small compared to unity, as was shown by Tartakovski [43]. In our case this condition is satisfied. An appreciable change in the amplitude or phase fluctuations can occur only in a distance of the order of the correlation distance in the medium, and since the wavelength is much smaller than the correlation distance, the change in a wavelength will be small.

Using cartesian coordinates with origin at the focus and abscissa along the principal axis of the lens, we have

$$\vec{F}\cdot\vec{r} = xx' + yy' + zz' , \tag{2}$$

where x,y,z are the coordinates of the element ds of the sphere, and x',y',z' are the coordinates of the observation point. In this case the amplitude and phase fluctuation on the sphere are to be given as functions of the cartesian coordinates, i.e. $A(x,y,z)$ and $S(x,y,z)$. The coordinates x,y,z are connected by the relation $x^2+y^2+z^2 = F^2$, so that only two of them are independent variables. Choosing y and z to be independent, we obtain

$$x = \sqrt{F^2 - (y^2+z^2)} .$$

In the case of a paraxial lens this relation can be replaced by the approximation

$$x = F - \frac{1}{2}\frac{y^2+z^2}{F} .$$

Then Eq. (2) can be written as follows

$$\vec{F}\cdot\vec{r} = \left(F - \frac{1}{2}\frac{y^2+z^2}{F}\right)\cdot x' + yy' + zz' . \tag{2a}$$

In spherical coordinates with the same origin and with the polar axis directed along the principal axis, we have

$$\vec{F}\cdot\vec{r} = Fr\cos\gamma = Fr\left[\cos\theta\cos\theta_o + \sin\theta\sin\theta_o\cos(\phi - \phi_o)\right] ,$$

where F,θ,ϕ are the coordinates of the element ds of the sphere, and r,θ_o,ϕ_o are the coordinates of the observation point. The amplitude and phase fluctuations on the sphere are to be given as functions of the angular variables, i.e. $A(\theta,\phi)$ and $S(\theta,\phi)$.

In what follows we confine ourselves to paraxial lenses. Then we can assume that the distribution of the amplitude and phase fluctuations on the exit spherical surface of the lens coincides with the distribution on the input plane surface, or, in other words, coincides with the distribution in the incident wave. Introducing the logarithmic amplitude level by the relation $A = A_o e^B$, we can write the Debye formula in the following final form

$$p = \frac{i}{\lambda F} e^{-ikF} A_o \int_s e^{B+iS} \exp\left[ik\frac{\vec{F}\cdot\vec{r}}{F}\right] ds , \qquad (3)$$

where A_o is the amplitude of the incident (unperturbed) wave.

30. Statistical Averaging. The fluctuations of amplitude B and phase S are random functions of time.. In order to find the average distribution, it is necessary to carry out statistical averaging over all possible values of B and S. In doing this, we must bear in mind that the readings of measuring instruments are usually proportional to $|p|^2$(to the square of the pressure in the acoustical case), so that $\overline{|p|^2}$ characterizes the mean distribution for these instruments. However, we begin by calculating the mean field $\overline{p}$, which in the first place is simpler, and which in the second place is also needed to calculate the fluctuations of the field near the focus.

It must be emphasized that in order to calculate the average of the factor e^{B+iS} it is not enough to know just first order moments $(\overline{B},\overline{S})$ and second order moments $(\overline{B^2},\overline{S^2})$; we must either know the infinite set of moments of all orders or the distribution function of the random variables B and S. It can be asserted that the distribution of the random variables B and S is normal. In fact, the entire distance which contributes to the values of the quantities B and S can be divided up into a large number of segments with lengths of the order of the

correlation distance of the refractive index of the medium, and the changes in the quantities B and S along different segments will be statistically independent. Therefore, the quantities B and S are made up of a large number of independent random variables. Then, according to the central limit theorem of probability theory, proved by Lyapunov [44], quantities like B and S obey the normal distribution law.

For the two dependent random variables B and S, the distribution law can be written as follows

$$f(B,S)\,dB\,dS = \frac{H}{\pi} \exp\left[-(aB^2 + 2bBS + cS^2)\right] dB\,dS, \tag{4}$$

where

$$H = \sqrt{ac - b^2}\,, \quad ac - b^2 > 0. \tag{5}$$

The parameters a,b,c of the distribution can easily be expressed in terms of the mean squares $\overline{B^2}$, $\overline{S^2}$ and the correlation coefficient R_{bs}. It follows from (4) that

$$\overline{B^2} = \frac{c}{2H^2}\,, \quad \overline{S^2} = \frac{a}{2H^2}\,, \quad \overline{BS} = -\frac{b}{2H^2}\,, \tag{6}$$

whence by (5) we find

$$\overline{B^2}\;\overline{S^2} - (\overline{BS})^2 = \frac{1}{4H^2} \quad \text{or} \quad \overline{B^2}\;\overline{S^2}\,(1 - R_{bs}^2) = \frac{1}{4H^2}\,.$$

Consequently

$$H = \frac{1}{2\sqrt{\overline{B^2}}\,\sqrt{\overline{S^2}}\,\sqrt{1 - R_{bs}^2}}\,. \tag{7}$$

Using (6) we obtain

$$a = \frac{1}{2\overline{B^2}\,(1 - R_{bs}^2)}\,, \quad c = \frac{1}{2\overline{S^2}\,(1 - R_{bs}^2)}\,, \quad b = -\frac{R_{bs}}{2\sqrt{\overline{B^2}}\sqrt{\overline{S^2}}\,(1 - R_{bs}^2)}\,. \tag{8}$$

The average of the quantity e^{B+iS} can now be calculated easily by using (4):

$$\overline{e^{B+iS}} = \iint\limits_{-\infty}^{+\infty} e^{B+iS} f(B,S)\,dB\,dS \tag{9}$$

$$= \frac{H}{\pi} \iint\limits_{-\infty}^{+\infty} e^{B+iS}\, e^{-(aB^2 + 2bBS + cS^2)}\,dB\,dS.$$

We write the double integral in the form

$$\int_{-\infty}^{+\infty} e^{iS}\, e^{-cS^2}\,dS \int_{-\infty}^{+\infty} e^{B}\, e^{-(aB^2 + 2bBS)}\,dB.$$

The integration over B gives

$$\sqrt{\frac{\pi}{a}}\ \exp\left[\frac{(1 - 2bS)^2}{4a}\right].$$

Then the double integral becomes

$$\sqrt{\frac{\pi}{a}} \int_{-\infty}^{+\infty} \exp\left[-(c - \frac{b^2}{a})S^2 - (\frac{b}{a} - i)S + \frac{1}{4a}\right] dS.$$

Since the parameters of the normal distribution satisfy the inequality $c - \left(b^2/a\right) > 0$, calculation of this integral gives

$$\frac{\pi}{H} \exp\left[\frac{1}{4}\,\frac{c - a - 2ib}{ac - b^2}\right].$$

Finally we obtain

$$\overline{e^{B+iS}} = \exp\left[\frac{1}{4}\,\frac{c - a - 2ib}{ac - b^2}\right].$$

It remains to substitute for the parameters a,b and c by using Eq. (8):

$$\overline{e^{B+iS}} = \exp\left[\frac{1}{2}\overline{B^2} - \frac{1}{2}\overline{S^2} + i\sqrt{\overline{B^2}}\sqrt{\overline{S^2}}\,R_{bs}\right]. \tag{11}$$

Thus, for the mean field we obtain

$$\overline{p} = \frac{i}{\lambda F}\, e^{-ikF} A_o \exp\left[\frac{1}{2}\overline{B^2} - \frac{1}{2}\overline{S^2} + i\sqrt{\overline{B^2}}\sqrt{\overline{S^2}}\,R_{bs}\right]\int_s e^{ik\vec{F}\cdot\vec{r}/F} ds\ . \tag{12}$$

However, as remarked, the instruments usually measure $\overline{|p|^2} = \overline{p^*p}$. Let us now calculate this quantity. By (3) we have

$$\overline{p^*\,p} = \frac{A_o^2}{\lambda^2F^2}\int_s\int_s \overline{\exp(B_1+B_2+i(S_1-S_2))}\,\exp\left[\frac{ik}{F}(\vec{F}_1-\vec{F}_2)\cdot\vec{r}\right]ds_1 ds_2. \tag{13}$$

It is easy to show that the factors $\exp(B_1+B_2)$ and $\exp[i(S_1-S_2)]$ are statistically independent. To do so we calculate the average of the product $(B_1+B_2)(S_1-S_2)$, obtaining

$$\overline{(B_1+B_2)(S_1-S_2)} = \overline{B_1S_1} - \overline{B_2S_2} + \overline{B_2S_1} - \overline{B_1S_2}\ .$$

Since any two points in the plane perpendicular to the direction of propagation of the wave are equivalent, we have

$$\overline{B_1S_1} = \overline{B_2S_2}\ , \quad \overline{B_2S_1} = \overline{B_1S_2}\ ,$$

and therefore

$$\overline{(B_1+B_2)(S_1-S_2)} = 0. \tag{14}$$

If the fluctuations are independent, the correlation coefficient vanishes. The converse statement is in general incorrect, i.e., vanishing of the correlation coefficient does not imply

statistical independence. However, if the fluctuations obey a normal law (are normally correlated), then the converse statement is also true. Therefore (14) implies the statistical independence of the fluctuations $B_1 + B_2$ and $S_1 - S_2$, despite the fact that the four quantities B_1, B_2, S_1 and S_2 are pairwise dependent.

Any functions of the independent fluctuations will also be statistically independent. In particular, this applies to the functions $\exp(B_1 + B_2)$ and $\exp[i(S_1 - S_2)]$. Therefore

$$\overline{e^{B_1+B_2}\, e^{i(S_1-S_2)}} = \overline{e^{B_1+B_2}}\;\overline{e^{i(S_1-S_2)}}. \tag{15}$$

For the difference in phase fluctuations $\xi = S_1 - S_2$ we write the distribution law in the following form

$$\phi(\xi)d\xi = \frac{1}{\sqrt{2\pi\,\overline{\xi^2}}} \exp\left(-\frac{\xi^2}{2\overline{\xi^2}}\right) d\xi. \tag{16}$$

By (16) we have

$$\overline{e^{i(S_1-S_2)}} = \frac{1}{\sqrt{2\pi\,\overline{\xi^2}}} \int_{-\infty}^{+\infty} e^{i\xi} \exp\left(-\frac{\xi^2}{2\overline{\xi^2}}\right) d\xi = \exp\left(-\frac{\overline{\xi^2}}{2}\right). \tag{17}$$

Moreover

$$\overline{\xi^2} = \overline{(S_1 - S_2)^2} = \overline{S_1^2} + \overline{S_2^2} - \overline{2S_1S_2} = 2\overline{S^2}\,(1 - R_s), \tag{18}$$

since $\overline{S_1^2} = \overline{S_2^2} = \overline{S^2}$. Therefore

$$\overline{e^{i(S_1-S_2)}} = e^{-\overline{S^2}(1-R_s)}. \tag{19}$$

Similarly, for the sum of the amplitudes $\eta = B_1 + B_2$ the distribution law has the form

$$\psi(\eta)d\eta = \frac{1}{\sqrt{2\pi\,\overline{\eta^2}}} \exp\left(-\frac{\eta^2}{2\overline{\eta^2}}\right) d\eta, \tag{20}$$

from which it follows that

$$\overline{e^{B_1+B_2}} = \frac{1}{\sqrt{2\pi\,\overline{\eta^2}}} \int_{-\infty}^{+\infty} e^{\eta} \exp\left(-\frac{\eta^2}{2\overline{\eta^2}}\right) d\eta = \exp \frac{\overline{\eta^2}}{2} . \tag{21}$$

Since

$$\overline{\eta^2} = \overline{(B_1 + B_2)^2} = \overline{B_1^2} + \overline{B_2^2} + \overline{2B_1B_2} = 2\overline{B^2}(1 + R_b), \tag{22}$$

we have

$$\overline{e^{B_1+B_2}} = e^{\overline{B^2}(1+R_b)} . \tag{23}$$

By (19) and (23), Eq. (13) becomes

$$\overline{p^*p} = \frac{A_o^2}{\lambda^2F^2}\, e^{\overline{B^2}-\overline{S^2}} \iint\limits_{s\ s} e^{\overline{B^2}R_b+\overline{S^2}R_s} \exp\left[i\,\frac{k}{F}(\vec{F}_1-\vec{F}_2)\cdot\vec{r}\right] ds_1 ds_2 . \tag{24}$$

Eq. (24) can be written in the following way:

$$\overline{p^*p} = \frac{A_o^2}{\lambda^2F^2}\, e^{2\overline{B^2}} \iint\limits_{s\ s} e^{\overline{B^2}(R_b-1)+\overline{S^2}(R_s-1)} \exp\left[i\,\frac{k}{F}(\vec{F}_1-\vec{F}_2)\cdot\vec{r}\right] ds_1 ds_2 . \tag{25}$$

Here the expression $A_o^2e^{2\overline{B^2}}$ represents the mean square amplitude of the wave incident on the surface of the lens. In fact, the amplitude A on the surface of the lens and the logarithmic amplitude level are connected by the relation $A = A_oe^B$ so that $A^2 = A_o^2e^{2B}$. Averaging this relation using the distribution

$$\psi(B)dB = \frac{1}{\sqrt{2\pi\,\overline{B^2}}} \exp\left(-\frac{B^2}{2\overline{B^2}}\right) dB,$$

we obtain

$$\overline{A^2} = A_o^2e^{2\overline{B^2}} .$$

Finally Eq. (25) can be written in the following way:

$$\overline{p^*p} = \frac{\overline{A^2}}{\lambda^2 F^2} \iint\limits_{s\ s} e^{\overline{B^2}(R_b-1)+\overline{S^2}(R_s-1)} \exp\left[i\ \frac{k}{F}(\vec{F}_1-\vec{F}_2)\cdot\vec{r}\right] ds_1 ds_2. \tag{26}$$

Now it is easy to obtain the formula for the mean square fluctuation of the field near the focus. In fact we have

$$\overline{|\Delta p|^2} = \overline{|p - \overline{p}|^2} = \overline{(p^* - \overline{p^*})(p - \overline{p})} = \overline{p^*p} - \overline{p^*}\ \overline{p}\ , \tag{27}$$

whence by (12) and (26) we obtain

$$\overline{|\Delta p|^2} = \frac{\overline{A^2}}{\lambda^2 F^2} \iint\limits_{s\ s} \left(e^{\overline{B^2}(R_b-1)+\overline{S^2}(R_s-1)} - e^{-\overline{B^2}-\overline{S^2}}\right) \exp\left[i\ \frac{k}{F}(\vec{F}_1-\vec{F}_2)\cdot\vec{r}\right] ds_1 ds_2. \tag{28}$$

It follows from (28) and (26) that not only the fluctuations but also the mean distribution depend on the amplitude and phase fluctuations in the incident wave.

It is necessary to make a remark concerning the formula $\overline{A^2} = A_o^2 e^{2\overline{B^2}}$, which implies that the energy flow in the incident wave increases with distance. Of course, this result contradicts the law of energy conservation, since there are no sources which might increase the energy in the incident wave as it propagates. The reason for this contradiction is contained in the method of small perturbations (both in the ordinary and modified forms), within the framework of which our investigation is carried out. The point is that for the zero order approximation we take the plane wave $\psi_o = kx$, on which the scattered waves are superimposed. Since attenuation of the plane wave is not taken into account, its energy is regarded as constant, and more and more energy from the scattered waves is added to this energy as the path traversed by the wave in the inhomogeneous medium increases. It is this which leads to the energy of the resultant wave increasing with distance. However, the scattering actually diminishes the energy of the primary wave, i.e., the regular wave field is converted into an irregular field

in such a way that the resultant flow remains constant and does not depend on the distance. We can make the theory agree with the law of energy conservation if we introduce the normalizing factor $e^{-\overline{B^2}}$ and set $p' = e^{-\overline{B^2}} p$ in the incident wave. Since the normalizing factor depends only on x, it attenuates all the waves emitted by different elements of the lens to the same extent and consequently does not affect the distribution behind the lens. Using (11) we obtain.

$$\overline{p'} = \exp\left(-\frac{1}{2}\overline{B^2} - \frac{1}{2}\overline{S^2} + i\sqrt{\overline{B^2}}\sqrt{\overline{S^2}}\,R_{bs}\right) ,$$

for the average normalized field in the incident wave, from which it follows that the mean field goes to zero as the distance increases. The transition to a normalized field is equivalent to changing the amplitude A to the amplitude A_o in Eqs. (26) and (28), which characterize the mean distribution and the distribution of fluctuations near the focus.

31. Various Special Cases. The basic formulas (26) and (28) of the preceding section were obtained without special assumptions concerning the form of the correlation coefficient for the refractive index and without the restriction that the amplitude and phase fluctuations in the incident wave be small. Suppose now that the refractive index correlation coefficient has the form $N = \exp(-r^2/a^2)$. Then it follows from Eqs. (219), (220), (225) and (226) of Part II that

$$\overline{B^2} + \overline{S^2} = \alpha \; , \tag{29}$$

$$\overline{B^2}(R_b - 1) + \overline{S^2}(R_s - 1) = \alpha(e^{-\ell^2/a^2} - 1), \tag{30}$$

where

$$\alpha = \sqrt{\pi}\,\overline{\mu^2}\,k^2 aL . \tag{31}$$

(It is interesting to note that in this calculation the terms containing the integral exponential function cancel each other.) The formula (26) for the mean distribution becomes

$$\overline{p^*p} = \frac{A_o^2}{\lambda^2 F^2} \int_S \int_S e^{\alpha[\exp(-\ell^2/a^2)-1]} \exp\left(i \frac{k}{F}(\vec{F}_1 - \vec{F}_2)\cdot\vec{r}\right) ds_1 ds_2. \tag{32}$$

If the amplitude and phase fluctuations in the incident wave are small ($\overline{B^2} \ll 1$, $\overline{S^2} \ll 1$), the exponential factors in (28) can be expanded in series so that

$$e^{\overline{B^2}(R_b-1)+\overline{S^2}(R_s-1)} - e^{-\overline{B^2}-\overline{S^2}} \sim \overline{B^2}R_b + \overline{S^2}R_s.$$

Then from Eq. (28) we obtain

$$\overline{|\Delta p|^2} = \frac{A_o^2}{\lambda^2 F^2} \int_S \int_S (\overline{B^2}R_b + \overline{S^2}R_s) \exp\left[i \frac{k}{F}(\vec{F}_1 - \vec{F}_2)\cdot\vec{r}\right] ds_1 ds_2. \tag{33}$$

If in addition we assume that the refractive index correlation coefficient is $N = \exp(-r^2/a^2)$, then Eq. (33) becomes

$$\overline{|\Delta p|^2} = \frac{A_o^2\alpha}{\lambda^2 F^2} \int_S \int_S e^{-\ell^2/a^2} \exp\left[i \frac{k}{F}(\vec{F}_1 - \vec{F}_2)\cdot\vec{r}\right] ds_1 ds_2. \tag{34}$$

Chapter VIII

THE MEAN DISTRIBUTION NEAR THE FOCUS

Finding the mean distribution near the focus when there are fluctuations in the incident wave is a more complicated mathematical problem than finding the distribution in the presence of regular amplitude and phase aberration. The complication stems from the necessity of calculating a four-fold integral (two integrations over the surface s) instead of a double integral. In order to solve the problem even for limiting cases, it is necessary from the very beginning to make the simplifying assumption that the lens is limited by a square diaphragm. It seems reasonable to suppose that the field near the focus of the lens will agree in its main features with the field of the lens when it is limited by a circular diaphragm. In considering a lens limited by a square diaphragm it is expedient to use a cartesian coordinate system with the abcissa along the principal axis of the lens. By (2a) we have

$$(\vec{F}_1 - \vec{F}_2)\cdot\vec{r} = (y_2^2 - y_1^2 + z_2^2 - z_1^2)\,\frac{x'}{2F} + (y_1 - y_2)\,y' + (z_1 - z_2)\,z'.$$

Since

$$\ell^2 = (y_1 - y_2)^2 + (z_1 - z_2)^2,$$

in cartesian coordinates Eq. (32) is written as

$$\overline{p^*p} = \frac{A_o^2}{\lambda^2F^2}\iiiint_{-h/2}^{h/2} \exp\left[\alpha\left(\exp\left[-\frac{(y_1-y_2)^2+(z_1-z_2)^2}{a^2}\right]-1\right)\right]\times \tag{35}$$

$$\times\exp\left(i\,\frac{k}{F}\left[(y_2^2-y_1^2+z_2^2-z_1^2)\frac{x'}{2F}+(y_1-y_2)y'+(z_1-z_2)z'\right]\right)\,dy_1dz_1dy_2dz_2,$$

where h is the dimension of the square diaphragm. We shall use Eq. (35) to study the distri-

bution both in the focal plane and along the principal axis of the lens.

32. The Mean Distribution in the Focal Plane. In the focal plane ($x' = 0$) Eq. (35) takes the form

$$\overline{p^*p} = \frac{A_o^2}{\lambda^2 F^2} \iiiint_{-h/2}^{h/2} \exp\left[-\alpha\left(1 - \exp\left[-\frac{(y_1-y_2)^2 + (z_1-z_2)^2}{a^2}\right]\right)\right] \times \tag{36}$$

$$\times \exp\left(i\frac{k}{F}[y'(y_1-y_2)+z'(z_1-z_2)]\right) dy_1 dz_1 dy_2 dz_2 \; .$$

The integrals which figure in this expresseion can be calculated in the two limiting cases of very small ($\alpha \ll 1$) and very large ($\alpha \gg 1$) fluctuations in the incident wave. In the first case $\alpha \ll 1$. If we set the first factor behind the integral equal to unity, then Eq. (36) simplifies greatly to

$$\overline{p^*p} = \frac{A_o^2}{\lambda^2 F^2} \iiiint_{-h/2}^{h/2} \exp\left[i\frac{k}{F}(y'y_1 - y'y_2 + z'z_1 - z'z_2)\right] dy_1 dy_2 dz_1 dz_2 . \tag{37}$$

The fourfold integral factors into single integrals, which are simply evaluated. Then we obtain the familiar formula for the intensity distribution in the focal plane of the lens in the absence of fluctuations in the incident wave:

$$\overline{p^*p} = A_o^2 \frac{h^4}{\lambda^2 F^2} \left(\frac{\sin \frac{hky'}{2F}}{\frac{hky'}{2F}}\right)^2 \times \left(\frac{\sin \frac{hkz'}{2F}}{\frac{hkz'}{2F}}\right)^2 \; . \tag{38}$$

The dimensions $\Delta y' = \Delta z'$ of the central focal spot are given by the formula

$$\Delta y' = \Delta z' = \frac{2\pi F}{kh} . \tag{39}$$

The intensity in the focus is given by the quantity

$$I_o = \frac{h^4}{\lambda^2 F^2} A_o^2. \tag{40}$$

The study of large fluctuations in the incident wave can give substantially new results only in the case where the dimensions of the lens are large or at least of the same order of magnitude as the correlation distance in the medium ($h \gtrsim a$). For if the dimensions of the lens are small compared to the correlation distance ($h \ll a$), then the amplitude and phase are identical at all points of the lens, and their fluctuations in time, even if large, have no effect at all on the diffraction image. Therefore, we consider the case where the two conditions

$$a \gg 1 \tag{41}$$

and

$$h \gtrsim a \tag{42}$$

are met. In this case we can use the following approximate representation of the exponential function:

$$\exp\left[- \frac{(y_1 - y_2)^2 + (z_1 - z_2)^2}{a^2}\right] \sim 1 - \frac{(y_1 - y_2)^2 + (z_1 - z_2)^2}{a^2} \tag{43}$$

For

$$(y_1 - y_2)^2 + (z_1 - z_2)^2 < a^2$$

two terms in the expansion give a good enough description of the exponential function. For

$$(y_1 - y_2)^2 + (z_1 - z_2)^2 \gtrsim a^2$$

the contribution to the integral (36) made by the exact function or by its approximate expression (43) are relatively small (the smaller the larger α). Therefore the form of the function for large values of the argument is not important.

By using (43) Eq. (36) takes the form

$$\overline{p^*p} = \frac{A_o^2}{\lambda^2 F^2} \iint_{-h/2}^{h/2} \exp\left[-\frac{\alpha}{a^2}(y_1 - y_2)^2 + i\,\frac{ky'}{F}(y_1 - y_2)\right] dy_1 dy_2 \times \tag{44}$$

$$\times \iint_{-h/2}^{h/2} \exp\left[-\frac{\alpha}{a^2}(z_1 - z_2)^2 + i\,\frac{kz'}{F}(z_1 - z_2)\right] dz_1 dz_2 .$$

We introduce the relative coordinate $y = y_1 - y_2$ and the center of mass coordinate $y_o = \frac{1}{2}(y_1 + y_2)$. In calculating the integral an important role is played by the values of y which satisfy the condition

$$\frac{\alpha}{a^2}\, y^2 \sim 1 \quad \text{or} \quad y^2 \sim \frac{a^2}{\alpha} .$$

For sufficiently large α

$$y^2 \ll a^2 .$$

Then by (42)

$$y^2 \ll h^2 \tag{45}$$

a fortiori. If this condition is met, we can integrate with respect to y in (44) between the limits $-\infty$ and $+\infty$. Then we obtain

$$\iint_{-h/2}^{h/2} \exp\left[-\frac{\alpha}{a^2}(y_1 - y_2)^2 + i\frac{ky'}{F}(y_1 - y_2)\right] dy_1 dy_2 = \tag{46}$$

$$\int_{-h/2}^{h/2} dy_o \int_{-\infty}^{+\infty} \exp\left[-\frac{\alpha}{a^2} y^2 + i \frac{ky'}{F} y\right] dy = ha \sqrt{\frac{\pi}{\alpha}} \exp\left[-\frac{1}{\alpha}\left(\frac{kay'}{2F}\right)^2\right] .$$

The second double integral in (44) is identical with the first. Therefore, we finally obtain

$$\overline{p^*p} = I \exp[-\beta(y'^2 + z'^2)], \tag{47}$$

where

$$I = \frac{\pi a^2 h^2}{\alpha \lambda^2 F^2} A_o^2, \tag{48}$$

$$\beta = \frac{1}{\alpha}\left(\frac{ka}{2F}\right)^2 . \tag{49}$$

Introducing the radial distance $r^2 = y'^2 + z'^2$ in the focal plane, we write (47) as

$$\overline{p^*p} = Ie^{-\beta r^2} . \tag{50}$$

Thus the intensity falls off monotonically as we go away from the focus. By (49) the effective radius of the focal spot $r_{eff} = 1/\sqrt{\beta}$ is

$$r_{eff} = \sqrt{\alpha}\ \frac{2F}{ka} , \tag{51}$$

i.e., it increases as the fluctuations increase, as is to be expected. The dependence on the focal distance and the wave number is the same as in the absence of fluctuations. However, the correlation distance takes the place of the dimension of the diaphragm. This is quite natural, since in the absence of fluctuations the amplitude and phase are constant over the area bounded by the diaphragm, while in the presence of fluctuations, the amplitude and phase are approximately constant over an area whose dimensions do not exceed the correlation distance.

The value of the intensity in the focus is given by the quantity I. As can be seen from (48), this quantity decreases as the fluctuations increase. Thus, as the fluctuations increase the effective radius of the focal spot increases, while the value of the intensity at

the maximum decreases. The spot becomes more "smeared out". However the total flow of energy through the focal plane remains equal to the flow of energy through the diaphragm of the lens. To confirm this, we calculate the total energy flow through the focal plane. It is proportional to the integral

$$\int_0^\infty \overline{p^*p}\; 2\pi r dr.$$

Substituting from (50) for $\overline{p^*p}$, we obtain

$$\int_0^\infty I e^{-\beta r^2}\; 2\pi r dr = \pi \frac{I}{\beta} .$$

Using (48) and (49) to replace I and β, we finally have

$$\int_0^\infty \overline{p^*p}\; 2\pi r dr = h^2 A_o^2 . \tag{52}$$

The right hand side of (52) is proportional to the flow of energy through the diaphragm of the lens. Thus, the flow of energy through the focal plane is equal to the flow through the diaphragm of the lens. Eq. (47), (48) and (49) satisfy the law of conservation of energy.

By (39) and (51) the ratio of the effective radius to the dimensions of the focal spot in the absence of fluctuations is given by the formula

$$\frac{r_{eff}}{\Delta y'} = \frac{\sqrt{\alpha}}{\pi} \frac{h}{a} . \tag{53}$$

For equal intensities in the incident wave, the ratio of the intensities at the focus obtained from (40) and (48) respectively is

$$\frac{I}{I_o} = \frac{\pi}{\alpha} \frac{a^2}{h^2} . \tag{54}$$

Finally, we consider a numerical example. Suppose that the dimensions of the lens are equal to the correlation distance (h = a). Moreover, let $\sqrt{\overline{S^2}} = \pi$, which corresponds (because of the statistical independence of phase fluctuations separated by a distance h = a) to a phase difference of 2π between the extreme rays incident on the lens, or to a one wavelength path difference between the extreme rays. Suppose further that the amplitude fluctuations are negligibly small compared to the phase fluctuations, i.e., $\overline{B^2} \ll \overline{S^2}$, which is true for the region of small values of the wave parameter. Then $\alpha = \overline{S^2} + \overline{B^2} \sim \overline{S^2} \sim 10$ and the conditions (41) and (42) for the applicability of the theory can be regarded as satisfied. Then Eqs. (53) and (54) give

$$\frac{r}{\Delta y'} \sim 1 \quad , \quad \frac{I}{I_o} \sim 0.3 \; .$$

The effective radius of the focal spot coincides with the dimensions of the focal spot in the absence of fluctuations. The intensity at the focus is 0.3 times the intensity at the focus in the absence of fluctuations. If the amplitude fluctuations cannot be neglected, then taking them into account increases the dimensions of the focal spot and decreases the intensity in the focus, as can be seen from Eqs. (53) and (54). Increasing the dimensions of the diaphragm leads to the same results.

33. The Mean Distribution along the Principal Axis. On the principal axis of the lens $y' = z' = 0$, and Eq. (35) becomes

$$\overline{p^*p} = \frac{A_o^2}{\lambda^2 F^2} \iiiint_{-h/2}^{h/2} \exp\left[\alpha\left(\exp\left[-\frac{(y_1-y_2)^2+(z_1-z_2)^2}{a^2}\right] - 1\right)\right] \times \tag{55}$$

$$\times \exp\left[i\,\frac{kx'}{2F^2}\,(y_2^2-y_1^2+z_2^2-z_1^2)\right] dy_1 dz_1 dy_2 dz_2 .$$

grals of the form

$$\int_{-h/2}^{h/2} \exp\left[i \frac{kx'}{2F^2} y_2^2\right] dy_2 = 2F\sqrt{\frac{2}{kx'}} \int_{0}^{\frac{1}{F}\sqrt{\frac{kx'}{2}}\frac{h}{2}} \exp\left[i\eta^2\right] d\eta =$$

$$= \sqrt{\frac{\pi}{2}}\, h \; \frac{C\left(\frac{h}{2F}\sqrt{\frac{kx'}{2}}\right) + iS\left(\frac{h}{2F}\sqrt{\frac{kx'}{2}}\right)}{\frac{h}{2F}\sqrt{\frac{kx'}{2}}}$$

where $C(z)$ and $S(z)$ are the Fresnel integrals. Therefore

$$\overline{p^*p} = A_o^2 \frac{\pi^2 h^4}{4\lambda^2 F^2} \; \frac{\left[C^2\left(\frac{h}{2F}\sqrt{\frac{kx'}{2}}\right) + S^2\left(\frac{h}{2F}\sqrt{\frac{kx'}{2}}\right)\right]^2}{\left(\frac{h}{2F}\sqrt{\frac{kx'}{2}}\right)^4} \; . \tag{56}$$

As we go away from the focus along the x' axis, the intensity decreases in an oscillatory fashion and approaches zero.

Let us consider the case of large fluctuations, when the conditions (41) and (42) are met. Using the approximation (43) for the exponential function, we transform (35) into the form

$$\overline{p^*p} = \frac{A_o^2}{\lambda^2 F^2} \iint_{-h/2}^{h/2} \exp\left[-\frac{\alpha}{a^2}(y_1-y_2)^2 + i\,\frac{kx'}{2F^2}(y_2^2-y_1^2)\right] dy_1 dy_2 \times \tag{56a}$$

$$\times \iint_{-h/2}^{h/2} \exp\left[-\frac{\alpha}{a^2}(z_1-z_2)^2 + i\,\frac{kx'}{2F^2}(z_2^2-z_1^2)\right] dz_1 dz_2 .$$

Going over to the relative coordinate $y = y_1 - y_2$ and the center of mass coordinate $y_o = \frac{1}{2}(y_1 + y_2)$, we can integrate with respect to y between the limits $-\infty$ and $+\infty$, in the way shown in the preceding section. The double integrals take the form

$$\int_{-h/2}^{h/2} dy_o \int_{-\infty}^{+\infty} \exp\left[-\frac{\alpha}{a^2} y^2 + i\frac{kx'}{F^2} y_o y\right] dy. \tag{57}$$

Consider first the evaluation of the inner integral*. Introduce the new variable

$$\eta = y - ib,$$

where

$$b = \frac{a^2}{\alpha} \frac{ky_o x'}{2F^2} . \tag{58}$$

Then the integral becomes

$$\exp(-\frac{\alpha}{a^2} b^2) \int_{-\infty - ib}^{+\infty - ib} \exp(-\frac{\alpha}{a^2}\eta^2)d\eta = \exp(-\frac{\alpha}{a^2} b^2) \int_{-\infty}^{+\infty} \exp(-\frac{\alpha}{a^2}\eta^2)d\eta$$

$$= a\sqrt{\frac{\pi}{\alpha}} \exp\left[-\left(\frac{akx'}{2\sqrt{\alpha}\, F^2}\right)^2 y_o^2\right]$$

(The integration along the line parallel to the real axis can be replaced by integration along the real axis.) Now we can easily carry out the integration with respect to y_o in (57):

$$a\sqrt{\frac{\pi}{\alpha}} \int_{-h/2}^{h/2} \exp\left[-\left(\frac{akx'}{2\sqrt{\alpha}\, F^2}\right)^2 y_o^2\right] dy_o = \frac{\pi a h}{2\sqrt{\alpha}} \frac{\Phi\left(\frac{kahx'}{4\sqrt{\alpha}\, F^2}\right)}{\frac{kahx'}{4\sqrt{\alpha}\, F^2}} ,$$

---

* See footnote on p. 101.

where

$$\Phi(z) = \frac{2}{\sqrt{\pi}} \int_{0}^{z} e^{-x^2} dx$$

is the probability integral. Eq. (56a) can finally be written as

$$\overline{p^* p} = \frac{\pi}{4} I \left[\frac{\Phi(\gamma x')}{\gamma x'} \right]^2 , \tag{59}$$

where I is defined by Eq. (48), i.e.

$$I = \frac{\pi a^2 h^2}{\alpha \lambda^2 F^2} A_o^2 \quad \text{and} \quad \gamma = \frac{kah}{4\sqrt{\alpha}\, F^2} \cdot$$

As the argument z increases, $\Phi(z)$ increases more slowly than z. Therefore the distribution (59) represents a function which falls off monotonically as we go away from the focus.

Chapter IX

FLUCTUATIONS BEHIND THE LENS

The theoretical study of the distribution of fluctuations in the diffraction image is an even more complicated problem than finding the mean distribution. Even in the case of small fluctuations in the incident wave the distribution of fluctuations can be found only in the two limiting cases $h \ll a$ and $h \gg a$. However, at the focus itself the fluctuation can be calculated for any value of the ratio between h and a. The next two sections are devoted to a study of these matters.

34. The Distribution of Fluctuations. By (34) the distribution of fluctuations in the focal plane is given by the formula

$$\overline{|\Delta p|^2} = \frac{A_o^2 \alpha}{\lambda^2 F^2} \int\int\int\int_{-h/2}^{h/2} \exp\left[-\frac{(y_1 - y_2)^2 + (z_1 - z_2)^2}{a^2}\right] \times$$
$$\times \exp\left[i \frac{k}{F} \left[y'(y_1 - y_2) + z'(z_1 - z_2)\right]\right] dy_1 dz_1 dy_2 dz_2. \tag{60}$$

If the dimensions of the diaphragm are small compared to the correlation distance ($h \ll a$), then the first exponential factor in the integrand can be set equal to unity, and the integral (60) reduces to the integral (37). The distribution of fluctuations resembles the mean distribution (38). If we take the ratio of the average fluctuation to the average intensity at the same point, we get

$$\frac{\overline{|\Delta p|^2}}{\overline{|p|^2}} = \alpha = \overline{B^2} + \overline{S^2} . \tag{61}$$

The fluctuation of the field in diffraction image depends not only on the amplitude fluctuation

$\overline{B^2}$ but also on the phase fluctuation $\overline{S^2}$ in the incident wave, and moreover in the same way. There is nothing surprising in this, since fluctuations of the field p (e.g. the pressure) are caused both by amplitude and phase fluctuations in the incident wave.

If the dimensions of the diaphragm are large compared to the correlation distance ($h \gg a$), then introducing as usual the relative coordinates y,z and center of mass coordinates y_o, z_o, we can extend the integrations with respect to y and z from $-\infty$ to ∞. From (60) we obtain

$$\overline{|\Delta p|^2} = \frac{A_o^2 \alpha}{\lambda^2 F^2} \int_{-h/2}^{h/2} dy_o \int_{-\infty}^{+\infty} \exp\left[-\frac{y^2}{a^2} + i\frac{ky'}{F}y\right] dy \times \tag{62}$$

$$\times \int_{-h/2}^{h/2} dz_o \int_{-\infty}^{+\infty} \exp\left[-\frac{z^2}{a^2} + i\frac{kz'}{F}z\right] dz.$$

The double integrals in (62) are the same as the integral (46) of the preceding section. The difference is only in the coefficient before y^2 in the exponential ($1/a^2$ instead of α/a^2). Therefore we immediately get

$$\overline{|\Delta p|^2} = Ce^{-\delta r^2}, \tag{63}$$

where

$$C = \frac{\pi\alpha a^2 h^2}{\lambda^2 F^2} A_o^2, \tag{64}$$

$$\delta = \left(\frac{ak}{2F}\right)^2. \tag{65}$$

The fluctuations fall off monotonically as we go away from the focus, and therefore the distribution of fluctuations is quite different from the mean distribution, which for small fluctuations is approximately described by the relation (38) and has an oscillatory character.

By (34) the distribution of small fluctuations along the principal axis is given by the formula

$$\overline{|\Delta p|^2} = \frac{A_o^2 \alpha}{\lambda^2 F^2} \int\!\!\int\!\!\int\!\!\int_{-h/2}^{h/2} \exp\left[-\frac{(y_1 - y_2)^2 + (z_1 - z_2)^2}{a^2}\right] \times \tag{66}$$

$$\times \exp\left[\frac{ikx'}{2F^2}(y_2^2 - y_1^2 + z_2^2 - z_1^2)\right] dy_1 dz_1 dy_2 dz_2 .$$

In the case $h \ll a$ the distribution of fluctuations along the axis is the same as the mean distribution and is given by Eq. (56), in which it is only necessary to introduce an extra factor of α. In the case $h \gg a$, the right hand side of (66) factors into integrals of the form (57) and we get a solution like (59), i.e.

$$\overline{|\Delta p|^2} = \frac{\pi}{4} C \left[\frac{\Phi(\epsilon x')}{\epsilon x'}\right]^2 , \tag{67}$$

where

$$C = \frac{\pi \alpha a^2 h^2}{\lambda^2 F^2} A_o^2 , \tag{68}$$

$$\epsilon = \frac{kah}{4F^2} . \tag{69}$$

The distribution of fluctuations is not the same as the mean distribution, which is approximately described by (56). As we go away from the focus, the fluctuations fall off monotonically, whereas the intensity in the mean distribution oscillates.

In the case of large fluctuations in the incident wave ($\alpha \gg 1$), we can neglect the term $\exp(-\overline{B^2} - \overline{S^2})$ in Eq. (28) compared to $\exp\left[\overline{B^2}(R_b - 1) + \overline{S^2}(R_s - 1)\right]$; then Eq. (28) agrees with Eq. (26) for the mean distribution. All that was said about the mean distribution for large fluctuations in the incident wave remains valid for the fluctuations near the focus.

35. Fluctuations at the Focus. By (33) small fluctuations at the focus are given by the following general formula:

$$\overline{|\Delta p|^2} = \frac{A_o^2}{\lambda^2 F^2} \int_s \int_s (\overline{B^2} R_b + \overline{S^2} R_s) ds_1 ds_2, \tag{70}$$

from which it follows that the fluctuations at the focus depend both on amplitude fluctuations and phase fluctuations in the incident wave. Since at the focus

$$\overline{|p|^2} = \frac{A_o^2 s^2}{\lambda^2 F^2},$$

we get

$$\frac{\overline{|\Delta p|^2}}{\overline{|p|^2}} = \frac{1}{s^2} \int_s \int_s (\overline{B^2} R_b + \overline{S^2} R_s) ds_1 ds_2 \tag{70a}$$

for the relative fluctuation. Instead of calculating the fluctuation of the field at the focus, we can calculate the fluctuations of amplitude and phase. Setting

$$p = a \exp\left[i(\phi - kF + \frac{\pi}{2})\right],$$

we can rewrite (1) in the following form:

$$ae^{i\phi} = \frac{1}{\lambda F} \int_s Ae^{iS} ds.$$

Equating the real and imaginary parts separately, we obtain

$$a \cos \phi = \frac{1}{\lambda F} \int_s A \cos S \, ds, \quad a \sin \phi = \frac{1}{\lambda F} \int_s A \sin S \, ds.$$

In the case of small fluctuations these become

$$a = \frac{1}{\lambda F} \int_s A ds, \tag{71}$$

$$a\phi = \frac{1}{\lambda F} \int_S ASds. \tag{72}$$

Averaging the relation (71), we obtain

$$\bar{a} = \frac{1}{\lambda F} \int_S \bar{A}ds = \frac{1}{\lambda F} \bar{A}s. \tag{73}$$

Denoting the fluctuations of amplitude at the focus and in the incident wave by $a' = a - \bar{a}$ and $A' = A - \bar{A}$ respectively, we find that

$$a' = \frac{1}{\lambda F} \int_S A'ds. \tag{74}$$

From (72) we get

$$\bar{a}\phi = \frac{\bar{A}}{\lambda F} \int_S Sds \tag{75}$$

with an accuracy up to quantities of the first order. Dividing (74) and (75) by (73) we find that

$$\frac{a'}{\bar{a}} = \frac{1}{s} \int \frac{A'}{\bar{A}} ds \text{ , } \phi = \frac{1}{s} \int_S Sds.$$

From this we obtain

$$\overline{\left(\frac{a'}{\bar{a}}\right)^2} = \frac{1}{s^2} \int_S \int_S \left(\frac{\overline{A'}}{\bar{A}}\right)^2 R_a ds_1 ds_2, \tag{76}$$

$$\overline{\phi^2} = \frac{1}{s^2} \int_S \int_S \overline{S^2} R_s ds_1 ds_2. \tag{77}$$

Eqs. (76) and (77) were obtained by Krasilnikov and Tatarski [10]. The fluctuation of the amplitude of the field at the focus is determined from the fluctuation of the amplitude of the incident wave, and the phase fluctuation at the focus is determined from the phase fluc-

tuation in the incident wave. Only small fluctuations at the focus have this property. The amplitude (or phase) fluctuation at any other point of the diffraction image depends both on the amplitude fluctuation and on the phase fluctuation in the incident wave.

By using (30), Eq. (70a) can be written as follows:

$$\frac{\overline{|\Delta p|^2}}{\overline{|p|^2}} = \frac{a}{h^4} \int\int\int\int_{-h/2}^{h/2} \exp\left[-\frac{(y_1 - y_2)^2 + (z_1 - z_2)^2}{a^2}\right] dy_1 dz_1 dy_2 dz_2. \tag{78}$$

This integral can be calculated without making any assumptions about the ratio between the dimensions of the diaphragm and the correlation distance. The problem reduces to calculating the integral

$$\int_{-h/2}^{h/2} dy_2 \int_{-(h/2)-y_2}^{(h/2)-y_2} \exp(-y^2/a^2)dy, \tag{79}$$

where $y = y_1 - y_2$ is the relative coordinate. Calculating the inner integral gives

$$\int_{-(h/2)-y_2}^{(h/2)-y_2} \exp(-\frac{y^2}{a^2})dy = \frac{\sqrt{\pi}}{2} a \left[\Phi\left(\frac{h}{2a} - \frac{y_2}{a}\right) + \Phi\left(\frac{h}{2a} + \frac{y_2}{a}\right)\right],$$

where Φ is the probability integral. Then the integral (79) becomes

$$a\frac{\sqrt{\pi}}{2} \int_{-h/2}^{h/2} \left[\Phi\left(\frac{h}{2a} - \frac{y_2}{a}\right) + \Phi\left(\frac{h}{2a} + \frac{y_2}{a}\right)\right] dy_2 =$$

$$= a \frac{\sqrt{\pi}}{2} \int_{0}^{h/2} \left[\Phi\left(\frac{h}{2a} - \frac{y_2}{a}\right) + \Phi\left(\frac{h}{2a} + \frac{y_2}{a}\right) \right] dy_2$$

$$+ a \frac{\sqrt{\pi}}{2} \int_{-h/2}^{0} \left[\Phi\left(\frac{h}{2a} - \frac{y_2}{a}\right) + \Phi\left(\frac{h}{2a} + \frac{y_2}{a}\right) \right] dy_2$$

$$= \sqrt{\pi}\, a \int_{0}^{h/2} \left[\Phi\left(\frac{h}{2a} - \frac{y_2}{a}\right) + \Phi\left(\frac{h}{2a} + \frac{y_2}{a}\right) \right] dy_2 .$$

Introducing the variable $\xi = \left(h/2a\right) - \left(y^2/a\right)$, we transform the first integral into the form

$$\sqrt{\pi}\, a^2 \int_{0}^{h/2a} \Phi(\xi) d\xi .$$

Introducing the variable $\eta = \left(h/2a\right) + \left(y^2/a\right)$, we transform the second integral into the form

$$\sqrt{\pi}\, a^2 \int_{h/2a}^{h/a} \Phi(\eta) d\eta = \sqrt{\pi}\, a^2 \int_{0}^{h/a} \Phi(\eta) d\eta - \sqrt{\pi}\, a^2 \int_{0}^{h/2a} \Phi(\eta) d\eta .$$

Consequently

$$\sqrt{\pi}\, a \int_{0}^{h/2} \left[\Phi\left(\frac{h}{2a} - \frac{y_2}{a}\right) + \Phi\left(\frac{h}{2a} + \frac{y_2}{a}\right) \right] dy_2 = \sqrt{\pi}\, a^2 \int_{0}^{h/a} \Phi(\eta) d\eta .$$

This integral can be found in tables [39]:

$$\int_{0}^{h/a} \Phi(\eta) d\eta = \frac{h}{a} \Phi\left(\frac{h}{a}\right) + \frac{\exp(-h^2/a^2) - 1}{\sqrt{\pi}} .$$

Finally, we obtain

$$\frac{\overline{|\Delta p|^2}}{\overline{|p|^2}} = \frac{\pi a}{(\frac{h}{a})^4} \left[\frac{h}{a} \Phi \left(\frac{h}{a}\right) + \frac{\exp(-h^2/a^2) - 1}{\sqrt{\pi}} \right]^2 \tag{80}$$

for the fluctuations at the focus. The graph of this function is shown in Fig. 13. As the dimensions of the diaphragm increase, the fluctuations decrease. This explains the averaging effect of a large lens. In particular, this result is supported by the fact that the twinkling of stars is much more noticeable to the naked eye than when observed with a large telescope.

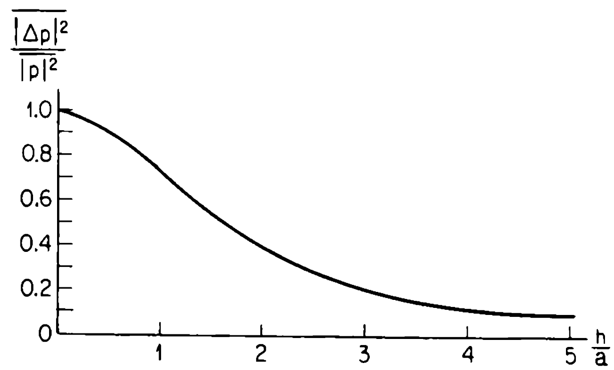

Fig. 13 Dependence of the relative fluctuation of the field amplitude at the focus on the size of the diaphragm.

Appendix I

It is apparent from Eqs. (99) and (100) of Part II that the phase fluctuations and the amplitude fluctuations are given by the real part and the imaginary part of the expression

$$\psi'(x,y,z) = -\frac{ik^2}{2\pi}\int_0^x\int_{-\infty}^{+\infty}\int \frac{1}{x-\xi}\exp\left[i\frac{k\rho^2}{2(x-\xi)}\right]\mu(\xi,\eta,\zeta)d\xi d\eta d\zeta\ , \tag{1}$$

respectively. We shall show that the exact formula (89) actually reduces to Eq. (1) in the case of large scale inhomogeneities.

We rewrite the exact formula as

$$\psi'(x,y,z) = -\frac{ik^2}{2\pi}\int_0^x\int_{-\infty}^{+\infty}\int \frac{1}{r}\exp\Big[ik\left[r-(x-\xi)\right]\Big]\ \mu(\xi,\eta,\zeta)d\xi d\eta d\zeta, \tag{2}$$

where

$$r = \sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}\ .$$

We assume that the refractive index fluctuations can be expanded in a Fourier integral

$$\mu(\xi,\eta,\zeta) = \int_{-\infty}^{+\infty}\int \phi(\xi,\kappa_2,\kappa_3)\ \exp\left[-i(\kappa_2\eta+\kappa_3\zeta)\right]d\kappa_2 d\kappa_3. \tag{3}$$

Then (2) can be rewritten as follows:

$$\psi'(x,y,z) = -\frac{ik^2}{2\pi}\int_0^x\int\int_{-\infty}^{+\infty}\int\int \frac{1}{r}\exp\Big[ik\left[r-(x-\xi)\right]\Big]\ \phi(\xi,\kappa_2,\kappa_3)\ \times \tag{4}$$

$$\times\ \exp\left[-i(\kappa_2\eta+\kappa_3\zeta)\right]d\xi d\eta d\zeta d\kappa_2 d\kappa_3 =$$

$$= - \frac{ik^2}{2\pi} \int_0^x \int\int\int\int_{-\infty}^{+\infty} \frac{1}{r} \exp(ikr) \exp\left(i\left[\kappa_2(y-\eta) + \kappa_3(z-\zeta)\right]\right) \times$$

$$\times \exp\left[-i(\kappa_2 y + \kappa_2 z) - ik(x-\xi)\right] \phi(\xi,\kappa_2,\kappa_3) d\xi d\eta d\zeta d\kappa_2 d\kappa_3 .$$

We introduce polar coordinates in the η,ζ plane by setting

$$y - \eta = \rho \cos \epsilon , \quad z - \zeta = \rho \sin \epsilon ,$$

$$\kappa_2 = \kappa \cos \alpha , \quad \kappa_3 = \kappa \sin \alpha .$$

Then

$$\kappa_2(y - \eta) + \kappa_3(z - \zeta) = \kappa\rho \cos(\epsilon - \alpha).$$

Integrating with respect to ϵ in (4) gives the Bessel function of order zero:

$$\psi' = - ik^2 \int_0^x d\xi \int\int_{-\infty}^{+\infty} \exp\left[-ik(x-\xi) - i(\kappa_2 y + \kappa_3 z)\right] \phi(\xi,\kappa_2,\kappa_3) d\kappa_2 d\kappa_3 \tag{5}$$

$$\times \int_0^\infty \frac{1}{r} \exp(ikr)\, I_0(\kappa\rho)\rho d\rho .$$

Using Sommerfeld's formula [45], we expand the spherical wave in cylindrical waves:

$$\frac{1}{r} \exp(ikr) = \int_0^\infty \frac{\lambda d\lambda}{\sqrt{\lambda^2 - k^2}}\, I_0(\lambda\rho) \exp\left[- \sqrt{\lambda^2 - k^2}\, (x-\xi)\right] \tag{6}$$

$$(x - \xi > 0).$$

Then the inner integral in (5) can be written as follows:

$$\int_0^\infty \rho d\rho \int_0^\infty \frac{\exp\left[-\sqrt{\lambda^2 - k^2}\,(x-\xi)\right]}{\sqrt{\lambda^2 - k^2}} I_o(\kappa\rho)\, I_o(\lambda\rho)\, \lambda d\lambda. \tag{7}$$

Because of the equality

$$f(\kappa) = \int_0^\infty \rho d\rho \int_0^\infty f(\lambda)\, I_o(\kappa\rho)\, I_o(\lambda\rho)\lambda d\lambda,$$

known as the Fourier-Bessel integral (valid in the case where $\int_0^\infty \kappa|f(\kappa)|d\kappa$ exists [27]), the double integral (7) equals

$$\frac{\exp\left[-\sqrt{\kappa^2 - k^2}\,(x-\xi)\right]}{\sqrt{\kappa^2 - k^2}}.$$

Finally we write Eq. (5) as follows:

$$\psi' = k^2 \int_0^x \int_{-\infty}^{+\infty}\int (k^2 - \kappa^2)^{-\frac{1}{2}} \exp\left[-i(\kappa_2 y + \kappa_3 z)\right] \times \tag{8}$$

$$\times \exp\left[-ik(x-\xi) + i(k^2 - \kappa^2)^{1/2}(x-\xi)\right] \phi(\xi,\kappa_2,\kappa_3)d\xi d\kappa_2 d\kappa_3.$$

Consider now the approximate formula (1). Using (3) to replace $\mu(\xi,\eta,\zeta)$, we obtain

$$\psi'(x,y,z) = -\frac{ik^2}{2\pi} \int_0^x \frac{d\xi}{x-\xi} \int_{-\infty}^{+\infty}\int \phi(\xi,\kappa_2,\kappa_3)d\kappa_2 d\kappa_3 \tag{9}$$

$$\times \int_{-\infty}^{+\infty} \exp\left[ik\frac{(y-\eta)^2}{2(x-\xi)} - i\kappa_2\eta\right]d\eta \int_{-\infty}^{+\infty} \exp\left[ik\frac{(z-\zeta)^2}{2(x-\xi)} - i\kappa_3\zeta\right]d\zeta.$$

We calculate the last two integrals by the saddle point method. To do this, we introduce new variables $\xi_1 = \xi/x$, $\eta_1 = \eta/x$, $y_1 = y/x$. Then the first of these integrals can be written as

$$x \int_{-\infty}^{+\infty} \exp\left(ikx\left[\frac{(y_1 - \eta_1)^2}{2(1-\xi_1)} - \frac{\kappa_2}{k}\eta_1\right]\right) d\eta_1. \tag{10}$$

Writing

$$f(\eta_1) \equiv i\left[\frac{(y_1 - \eta_1)^2}{2(1 - \xi_1)} - \frac{\kappa_2}{k}\eta_1\right] \tag{11}$$

and bearing in mind that the parameter kx is large ($kx \gg 1$), we use the saddle point method [27] to evaluate the integral

$$I = \int_{-\infty}^{+\infty} \exp[kxf(\eta_1)]d\eta_1. \tag{12}$$

We find the saddle point from the condition $\partial f/\partial\eta_1 = 0$, i.e.

$$\frac{y_1 - \eta_1^o}{1 - \xi_1} + \frac{\kappa_2}{k} = 0,$$

whence

$$\eta_1^o = \frac{\kappa_2}{k}(1 - \xi_1) + y_1. \tag{13}$$

Furthermore

$$\left(\frac{\partial^2 f}{\partial\eta_1^2}\right)_{\eta_1^o} = \frac{i}{1 - \xi_1}.$$

Making a series expansion of the function $f(\xi_1, \eta_1)$ near the saddle point, we obtain

$$f(\eta_1) = f(\eta_1^o) + \frac{1}{2}\left(\frac{\partial^2 f}{\partial\eta_1^2}\right)_{\eta_1^o}\Delta\eta_1^2 = -\frac{i}{2}\frac{\kappa_2^2}{k^2}(1-\xi_1) - i\frac{\kappa_2}{k}y_1 + \frac{i}{2}\frac{1}{1-\xi_1}\Delta\eta_1^2. \tag{14}$$

We must choose the path through the saddle point so that

$$\Delta\eta = (1 + i)s,$$

where s is a real parameter, Then

$$f(\eta_1) = -\frac{i}{2}\frac{\kappa_2^2}{k^2}(1 - \xi_1) - i\frac{\kappa_2}{k}y_1 - \frac{1}{1-\xi_1}s^2, \tag{15}$$

$$I = \sqrt{2i}\,\exp\left[-ix\frac{\kappa_2^2}{k}(1 - \xi_1) - i\kappa_2 y\right]\int_{-\infty}^{+\infty}\exp\left[-\frac{kx}{1-\xi_1}s^2\right]ds \tag{16}$$

$$= \sqrt{\frac{2\pi i(1 - \xi_1)}{kx}}\,\exp\left[-\frac{i}{2}x\frac{\kappa_2^2}{k}(1 - \xi_1) - i\kappa_2 y\right].$$

Eq. (9) then takes the form

$$\psi'(x,y,z) = k\int_0^x\int_{-\infty}^{+\infty}\int \exp\left[-i(\kappa_2 y + \kappa_3 z) - i\frac{1}{2k}(\kappa_2^2 + \kappa_3^2)(x - \xi)\right] \times \tag{17}$$

$$\times\, \phi(\xi,\kappa_2,\kappa_3)d\xi d\kappa_2 d\kappa_3.$$

It is now clear that in the case of large scale inhomogeneities ($k \gg x$) the exact formula (8) reduces to the approximate formula (17). To see this, it suffices to set $(k^2 - \kappa^2)^{-1/2} \sim k^{-1}$ in the amplitude and to take only the first two terms of the expansion $(k^2 - \kappa^2)^{1/2} \sim k(1 - \frac{1}{2}\frac{\kappa^2}{k^2})$ in the phase.

<u>Appendix II</u>

To evaluate certain integrals appearing in the text (Chapters 5 and 6) it is necessary to use the identity

$$\frac{1}{4\pi^2 S_1 S_2} \int\!\!\int_{-\infty}^{+\infty} \exp\left[-\frac{(y_1+y)^2+(z_1+z)^2}{2S_1}\right] \times \tag{1}$$

$$\times \exp\left[-\frac{(y_2-y_1)^2+(z_2-z_1)^2}{2S_2}\right] dydz = \frac{1}{2\pi(S_1+S_2)} \times$$

$$\times \exp\left[-\frac{(y_1+y_2)^2+(z_1+z_2)^2}{2(S_1+S_2)}\right]$$

which is well known in the theory of probability [12]. (The result of convolving two Gaussian distributions is again a Gaussian distribution.) First setting

$$y_1 = y_2 = \frac{\eta}{2}\ ,\ z_1 = \frac{\zeta}{2}\ ,\ z_2 = \frac{\zeta}{2} + \ell'\ ,\ S_1 = ia_1\ ,\ S_2 = -ia_2$$

in this identity, and then

$$y_1 = y_2 = \frac{\eta}{2}\ ,\ z_1 = \frac{\zeta}{2}\ ,\ z_2 = \frac{\zeta}{2} + \ell'\ ,\ S_1 = ia_1\ ,\ S_2 = ia_2\ ,$$

we can write the following two formulas:

$$\frac{1}{4\pi^2 a_1 a_2} \int\!\!\int_{-\infty}^{+\infty} \exp\left[-\frac{(\frac{\eta}{2}+y)^2+(\frac{\zeta}{2}+z)^2}{2ia_1}\right] \times \tag{2}$$

$$\times \exp\left[-\frac{(\frac{\eta}{2}-y)^2+(\frac{\zeta}{2}+\ell'-z)^2}{-2ia_2}\right]dydz = \frac{1}{2\pi i(a_1-a_2)}\times$$

$$\times \exp\left[-\frac{\eta^2+(\zeta+\ell')^2}{2i(a_1-a_2)}\right],$$

$$-\frac{1}{4\pi^2 a_1 a_2}\iint_{-\infty}^{+\infty}\exp\left[-\frac{(\frac{\eta}{2}+y)^2+(\frac{\zeta}{2}+z)^2}{2ia_1}\right]\times \tag{3}$$

$$\times \exp\left[-\frac{(\frac{\eta}{2}-y)^2+(\frac{\zeta}{2}+\ell'-z)^2}{2ia_2}\right]dydz = \frac{1}{2\pi i(a_1+a_2)}\times$$

$$\times \exp\left[-\frac{\eta^2+(\zeta+\ell')^2}{2i(a_1+a_2)}\right].$$

Going over from exponential functions to trigonometric functions, we obtain

$$\frac{1}{4\pi^2 a_1 a_2}\iint\left(\cos\left[\frac{(\frac{\eta}{2}+y)^2+(\frac{\zeta}{2}+z)^2}{2a_1}\right]\times\right. \tag{4}$$

$$\times\cos\left[\frac{(\frac{\eta}{2}-y)^2+(\frac{\zeta}{2}+\ell'-z)^2}{2a_2}\right] + \sin\left[\frac{(\frac{\eta}{2}+y)^2+(\frac{\zeta}{2}+z)^2}{2a_1}\right]\times$$

$$\left.\times\sin\left[\frac{(\frac{\eta}{2}-y)^2+(\frac{\zeta}{2}+\ell'-z)^2}{2a_2}\right]\right) dydz + \frac{1}{4\pi^2 a_1 a_2}\times$$

$$\times \iint \left(- i \cos\left[\frac{(\frac{\eta}{2} + y)^2 + (\frac{\zeta}{2} + z)^2}{2a_1}\right] \times \right.$$

$$\times \sin\left[\frac{(\frac{\eta}{2} - y)^2 + (\frac{\zeta}{2} + \ell' - z)^2}{2a_2}\right] + i \sin\left[\frac{(\frac{\eta}{2} + y)^2 + (\frac{\zeta}{2} + z)^2}{2a_1}\right] \times$$

$$\left. \times \cos\left[\frac{(\frac{\eta}{2} - y)^2 + (\frac{\zeta}{2} + \ell' - z)^2}{2a_2}\right] \right) dydz = \frac{1}{2\pi(a_1 - a_2)} \times$$

$$\times \sin \frac{\eta^2 + (\xi + \ell')^2}{2(a_1 - a_2)} - i \frac{1}{2\pi(a_1 - a_2)} \cos \frac{\eta^2 + (\zeta + \ell')^2}{2(a_1 - a_2)} ,$$

$$- \frac{1}{4\pi^2 a_1 a_2} \iint \left(\cos\left[\frac{(\frac{\eta}{2} + y)^2 + (\frac{\zeta}{2} + z)^2}{2a_1}\right] \times \right.$$

$$\times \cos\left[\frac{(\frac{\eta}{2} - y)^2 + (\frac{\zeta}{2} + \ell' - z)^2}{2a_2}\right] - \sin\left[\frac{(\frac{\eta}{2} + y)^2 + (\frac{\zeta}{2} + z)^2}{2a_1}\right] \times$$

$$\left. \times \sin\left[\frac{(\frac{\eta}{2} - y)^2 + (\frac{\zeta}{2} + \ell' - z)^2}{2a_2}\right] \right) dydz - \frac{1}{4\pi^2 a_1 a_2} \times$$

$$\times \iint \left(i \cos\left[\frac{(\frac{\eta}{2} + y)^2 + (\frac{\zeta}{2} + z)^2}{2a_1}\right] \times \right.$$

$$\times \sin\left[\frac{(\frac{\eta}{2} - y)^2 + (\frac{\zeta}{2} + \ell' - z)^2}{2a_2}\right] + i \sin\left[\frac{(\frac{\eta}{2} + y)^2 + (\frac{\zeta}{2} + z)^2}{2a_1}\right] \times$$

$$\times \cos\left[\frac{(\frac{\eta}{2} - y)^2 + (\frac{\zeta}{2} + \ell' - z)^2}{2a_2}\right]\Bigg) dydz = \frac{1}{2\pi(a_1 + a_2)} \times$$

$$\times \sin\left[\frac{\eta^2 + (\zeta + \ell')^2}{2(a_1 + a_2)}\right] - i \frac{1}{2\pi(a_1 + a_2)} \cos\left[\frac{\eta^2 + (\zeta + \ell')^2}{2(a_1 + a_2)}\right] .$$

Adding these equations and equating the real and imaginary parts separately, we find

$$\frac{2}{4\pi^2 a_1 a_2} \iint \sin\left[\frac{(\frac{\eta}{2} + y)^2 + (\frac{\zeta}{2} + z)^2}{2a_1}\right] \times \tag{6}$$

$$\times \sin\left[\frac{(\frac{\eta}{2} - y)^2 + (\frac{\zeta}{2} + \ell' - z)^2}{2a_2}\right] dydz = \frac{1}{2\pi(a_1 - a_2)} \times$$

$$\times \sin\left[\frac{\eta^2 + (\zeta + \ell')^2}{2(a_1 - a_2)}\right] + \frac{1}{2\pi(a_1 + a_2)} \sin\left[\frac{\eta^2 + (\zeta + \ell')^2}{2(a_1 + a_2)}\right] ,$$

$$\frac{2}{4\pi^2 a_1 a_2} \iint \cos\left[\frac{(\frac{\eta}{2} + y)^2 + (\frac{\zeta}{2} + z)^2}{2a_1}\right] \times \tag{7}$$

$$\times \sin\left[\frac{(\frac{\eta}{2} - y)^2 + (\frac{\zeta}{2} + \ell' - z)^2}{2a_2}\right] dydz = \frac{1}{2\pi(a_1 - a_2)} \times$$

$$\times \cos\left[\frac{\eta^2 + (\zeta + \ell')^2}{2(a_1 - a_2)}\right] + \frac{1}{2\pi(a_1 + a_2)} \cos\left[\frac{\eta^2 + (\zeta + \ell')^2}{2(a_1 + a_2)}\right] .$$

Subtracting (5) from (4), we obtain

$$\frac{2}{4\pi^2 a_1 a_2} \iint \cos\left[\frac{(\frac{\eta}{2} + y)^2 + (\frac{\zeta}{2} + z)^2}{2a_1}\right] \times$$

$$\times \cos\left[\frac{(\frac{\eta}{2} - y)^2 + (\frac{\zeta}{2} + \ell' - z)^2}{2a_2}\right] dydz = \frac{1}{2\pi(a_1 - a_2)} \times$$

$$\times \sin\left[\frac{\eta^2 + (\zeta + \ell')^2}{2(a_1 - a_2)}\right] - \frac{1}{2\pi(a_1 + a_2)} \sin\left[\frac{\eta^2 + (\zeta + \ell')^2}{2(a_1 + a_2)}\right],$$

$$\frac{2}{4\pi^2 a_1 a_2} \iint \sin\left[\frac{(\frac{\eta}{2} + y)^2 + (\frac{\zeta}{2} + z)^2}{2a_1}\right] \times$$

$$\times \cos\left[\frac{(\frac{\eta}{2} - y)^2 + (\frac{\zeta}{2} + \ell' - z)^2}{2a_2}\right] dydz = \frac{1}{2\pi(a_1 + a_2)} \times$$

$$\times \cos\left[\frac{\eta^2 + (\zeta + \ell')^2}{2(a_1 + a_2)}\right] - \frac{1}{2\pi(a_1 - a_2)} \cos\left[\frac{\eta^2 + (\zeta + \ell')^2}{2(a_1 - a_2)}\right].$$

In the notation of Section 18 of Part II, Eqs. (6) - (9) can be written as follows:

$$\iint_{-\infty}^{+\infty} \Phi_1\left[a_1, \sqrt{(\tfrac{\eta}{2} + y)^2 + (\tfrac{\zeta}{2} + z)^2}\right] \times$$

$$\times \Phi_1\left[a_2, \sqrt{(\tfrac{\eta}{2} - y)^2 + (\tfrac{\zeta}{2} + \ell' - z)^2}\right] dydz =$$

$$= \frac{1}{2}\left[\Phi_1(a_1 - a_2, \rho) + \Phi_1(a_1 + a_2, \rho)\right],$$

$$\int_{-\infty}^{+\infty}\!\!\int \Phi_2\left[a_1, \sqrt{\left(\frac{\eta}{2} + y\right)^2 + \left(\frac{\zeta}{2} + z\right)^2}\right] \times \tag{11}$$

$$\times\, \Phi_1\left[a_2 \sqrt{\left(\frac{\eta}{2} - y\right)^2 + \left(\frac{\zeta}{2} + \ell' - z\right)^2}\right] dydz =$$

$$= \frac{1}{2}\left[\Phi_2(a_1 - a_2, \rho) + \Phi_2(a_1 + a_2, \rho)\right],$$

$$\int_{-\infty}^{+\infty}\!\!\int \Phi_2\left[a_1, \sqrt{\left(\frac{\eta}{2} + y\right)^2 + \left(\frac{\zeta}{2} + z\right)^2}\right] \times \tag{12}$$

$$\times\, \Phi_2\left[a_2, \sqrt{\left(\frac{\eta}{2} - y\right)^2 + \left(\frac{\zeta}{2} + \ell' - z\right)^2}\right] dydz =$$

$$= \frac{1}{2}\left[\Phi_1(a_1 - a_2, \rho) - \Phi_1(a_1 + a_2, \rho)\right],$$

$$\int_{-\infty}^{+\infty}\!\!\int \Phi_1\left[a_1, \sqrt{\left(\frac{\eta}{2} + y\right)^2 + \left(\frac{\zeta}{2} + z\right)^2}\right] \times \tag{13}$$

$$\times\, \Phi_2\left[a_2, \sqrt{\left(\frac{\eta}{2} - y\right)^2 + \left(\frac{\zeta}{2} + \ell' - z\right)^2}\right] dydz =$$

$$= \frac{1}{2}\left[\Phi_2(a_1 + a_2, \rho) - \Phi_2(a_1 - a_2, \rho)\right],$$

where

$$\rho^2 = \eta^2 + (\zeta + \ell')^2. \tag{14}$$

In studying the problem of amplitude and phase fluctuations and the correlation between them at the receiving point, we must set $\ell' = 0$. In studying the transverse autocorrelation of amplitude (or phase) fluctuations at different points we have $\ell' \neq 0$.

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